Dois Problemas em Grafos

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I - What is a sparse graph?

Motivation: Integration of delta-weight mesh

Delta-weight mesh:

- A symmetric connected directed graph (V_G, E_G).
- $\delta : E_G \to \mathbb{R}$ (edge deltas).
- $w : E_G \to \mathbb{R}$ (edge weights), symmetric.

Assume:

- $\textsc{org}\, e$, $\textsc{dst}\, e$ are the endpoints of edge e.
- $\textsc{sym}\, e$ is the unique inverse of edge e.
- δ is antisymmetric: $\delta[\textsc{sym}\, e] = -\,\delta[e]$.
- w is symmetric and positive: $w[\textsc{sym}\, e] = w[e] > 0$.

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Terrain interpretation:

- Each vertex v is a point on the map.
- The height of the terrain at v is $z[v]$.
- $\delta[e]$ is a measurement of $z[\textsc{dst}\, e] - z[\textsc{org}\, e]$.
- Reliability of $\delta[e]$ is $w[e]$.
- Loops are irrelevant, can be deleted.
- Parallel edges can be condensed to make G simple.

Can be interpreted also as electrical circuit, spring network, . . .

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Problem: given G, compute z.

- Determined only up to an additive constant.
- If G is a tree, ignore w, add δ along paths.
- If G is not a tree, δ is usually inconsistent.

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Weighted least-squares solution: satisfies the *vertex equilibrium equations*.

$$z[u] = \frac{\displaystyle\sum_{e \in E_G[u]} w[e](z[\textsc{dst}\, e] - \delta[e])}{\displaystyle\sum_{e \in E_G[u]} w[e]} \tag{1}$$

Solving the equilibrium system

System $Az = b$ with $n = \#V_G$ equations and unknowns.

Matrix A has $O(m)$ nonzero elements, $m = \#E_G$.

Gaussian elimination: cost $O(nm^{0.5})$ ($O(n^{1.5})$ if planar).

Gauss-Seidel iteration: $O(m)$ per iteration ($O(n)$ if planar)...
...but needs at least $\Omega(n)$ iterations, sometimes $\Omega(n^2)$ ...
...so the total cost is $\Omega(nm)$ to $\Omega(n^2m)$ ($\Omega(n^2)$ to $\Omega(n^3)$).

Solving the equilibrium system

Multiscale algorithm [Saracchini and Stolfi 2011]: $\textsc{Solve}(G, \delta, w)$ returns (z)

1. If $V_G = \{v\}$, set $z[v] \leftarrow 0$, return z.
2. Find maximal independent set $R \subseteq V_G$ of max degree g.
3. $(G', \delta', w') \leftarrow \textsc{RemoveAndPatch}(R, G, \delta, w)$.
4. $z \leftarrow \textsc{Solve}(G', \delta', w')$.
5. For all $u \in R$, set $z[u] \leftarrow \textsc{Equilibrium}(u, G, \delta, w)$.
6. $z \leftarrow \textsc{GaussSeidel}(z, G, \delta, w)$.
7. Return z.

Solving the equilibrium system

Analysis of multiscale SOLVE for PLANAR graph G:

- Step 2: Cost $O(n)$.
- Step 3: Cost $O(n)$.
- $\#V'_G \leq \beta \#V_G$ for some $\beta < 1$.
- G' is planar.
- Step 4: Cost $O(n)$ by induction.
- Steps 6: $O(1)$ iterations, cost $O(m) = O(n)$.

Total cost: $O(m) = O(n)$!

Extension to non-planar delta-weight meshes

We need a family of graphs $\mathcal{F}$, where every G:

- is connected.
- is sparse ($m \leq An - B$ if $n \geq n_0$).
- has a minimum percentage of vertices of degree $\leq g$.
- admits a $\textsc{RemoveAndPatch}$ operation that preserves $\mathcal{F}$.
- includes the regular 3D meshes with holes.

What could that family be?

II - Characterizing triangulations

Characterizing triangulable graphs

Informal statement:

- A *2-triangulation* is a graph G drawn on a compact surface S (possibly with borders) in such a way that every face is a triangle.
- A graph G is *2-triangulable* if it admits a 2-triangulation.
- When is a graph 2-triangulable?

Every graph can be drawn on some surface, but the faces are not always triangles.

Formal (almost) definition statement:

A 2-triangulation is a triple $G = (V, E, T)$ where

- (V, E) is a simple undirected graph.
- Each $t \in T$ is incident to 3 distinct edges and 3 distinct vertices.
- Each $e \in E$ is incident to either one or two triangles.
- Each $v \in V$ is incident to at least one edge.
- Two triangles share a vertex only as a result of sharing edges.

The last condition ensures that the union of all triangles is a proper surface.

Simple examples:

- K_3 is triangulable as a sphere or as a disk.
- K_4 is triangulable as a sphere or as a disk.
- K_5 is triangulable as a Möbius strip.
- K_6 is triangulable as a Projective plane or as a Möbius strip.

Is K_7 2-triangulable?

Extension to d dimensions:

A d-triangulation is a tuple $G = (T_0, T_1, \ldots, T_d)$ where

- Each $t \in T_k$ has a boundary which is a k-simplex of G.
- Each $t \in T_k$, $k < d$, is incident to some element of T_d.
- Each $t \in T_{d-1}$ is incident to at most two elements T_d.
- Two elements of T_d share a vertex only as a result of sharing facets.

The last condition ensures that the union of all T_d is a d-dimensional pseudo-manifold with border.

The last condition may be strengthened to ensure a d-manifold with border (but the problem becomes hard for larger d).

Créditos das imagens