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Generating Functions for Counting

$$\overbrace{(1+x_1)(1+x_2)\cdots(1+x_k)}^{k} = \sum_{i=0}^{k}\binom{k}{i}x^i$$

$$2^k\left\{\begin{array}{l} 111\cdot 1 \quad + \\ x11111 \quad + \\ 1x1111 \quad + \\ \quad\vdots \\ \qquad + \\ x\cdots x \end{array}\right.$$

How many ways to select of sequence, n, of pennies + nickels that sum to k¢

$(x+x^5)(x+x^5)\cdots(x+x^5)$

answer is # terms of degree k

choose
nickel, nickel, penny, penny, nickel

$x^5 \quad x^5 \quad x \quad x \quad x^5 = x^{17}$

coefficient of x^k in

$(x+x^5)^n$

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k kinds of donuts, want to buy n donuts, how many such selections?

$$\binom{n+(k-1)}{k-1}$$

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chocolate $1 + 1\cdot c + 1\cdot c^2 + 1\cdot c^3 + \cdots$

vanilla $1 + 1\cdot v + 1\cdot v^2 + 1\cdot v^3 + \cdots$

k^{th} kind $1 + 1\cdot d + 1 d^2 + \cdots$

answer is the ~~coefficient of~~ number of degree n terms in

$(1+c+c^2+\cdots)(1+v+v^2+\cdots)\cdots(1+d+d^2+\cdots)$

$(1+x+x^2+\cdots)(1+x+x^2+\cdots)\quad(1+x+x^2+\cdots)$

coeff of x^n is # ways of selecting n donuts among k kinds

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coeff. of x^n in

$$\left(1+x+x^2+\cdots\right)^k = \left(\frac{1}{1-x}\right)^k = \frac{1}{(1-x)^k}$$

is $\displaystyle\binom{n+k-1}{k-1}$

Taylor series

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$

$$f_0 = F(0)$$

$$F'(x) = f_1 + 2f_2 x + 3f_3 x^2 + \cdots$$

$$f_1 = F'(0)$$

$$F''(x) = 2f_2 + 3\cdot 2 f_3 x + 4\cdot 3 f_4 x^2 \cdots$$

$$f_2 = \frac{F^{(2)}(0)}{2!}$$

$$f_n = \frac{F^{(n)}(0)}{n!}$$

$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \ldots$

$B(x) = b_0 + b_1 x + \cdots + b_n x^n + \ldots$

Convolution Counting Principle

$C(x) = A(x)B(x)$ # coefficient of x^n

tells me the # ways to form a mixed collection of a's & b's that totals to n

$$c_n = \underbrace{b_0 a_n + b_1 a_{n-1} + b_2 a_{n-2} + \cdots + b_n a_0}_{\text{convolution}}$$

Counting baskets of n fruits

in a basket must have

≤ 2 Bananas $\qquad B(x) = 1 + x + x^2 = \frac{1-x^3}{1-x}$

≤ 4 Pears $\qquad P(x) = 1 + x + x^2 + x^3 + x^4$

$$= \frac{1-x^5}{1-x}$$

even # apples $\qquad A(x) = 1 + 0x + 1x^2 + 0x^3 + 1 \cdot x^4 + \cdots$

$$= (1 + x^2 + x^4 + x^6 + \cdots)$$

$$= \frac{1}{1-x^2}$$

oranges is divisible by 5 $\qquad O(x) = (1 + x^5 + x^{10} + \cdots)$

$$= \frac{1}{1-x^5}$$

baskets of n fruit is coefficient of x^n

in $F(x) = B(x) \cdot P(x) \cdot A(x) \cdot O(x)$

$$= \frac{1-x^3}{1-x} \cdot \frac{\cancel{1-x^5}}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{\cancel{1-x^5}}$$

$$= \frac{1-x^3}{(1-x)^3 \cdot (1+x)}$$

$$F(x) = \frac{1-x^3}{(1-x)^3(1+x)}$$

$$= \frac{a}{1-x} + \frac{b}{(1-x)^2} + \frac{c}{(1-x)^3} + \frac{d}{1+x}$$

$$1-x^3 = a(1-x)^2(1+x) + b(1-x)(1+x)$$
$$c(1+x) + d(1-x)^3$$

let $x=-1$: $2 = d2^3 \Longrightarrow d = \frac{1}{4}$

$x=1$: $0 = c \cdot 2 \Longrightarrow c = 0$

$x=0$: $a+b+c+d = 1 \Longrightarrow a+b = \frac{3}{4}$

$x = \frac{1}{2}$: $\Longrightarrow b = \frac{3}{2}$, $a = -\frac{3}{4}$

Coeff of x^n : $a + b\binom{n+2-1}{2-1} + c(\cdot) + d(-1)^n$

$$-\frac{3}{4} + \frac{3}{2}\underbrace{\binom{n+1}{1}}_{n+1} + \frac{1}{4}(-1)^n$$

$n=1$?

$$= \frac{6(n+1) - 3 + (-1)^n}{4}$$

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