Mathematics for Computer Science

MIT 6.042J/18.062J

Stirling's formula, Asymptotics

Closed form for n!

Factorial defines a product:

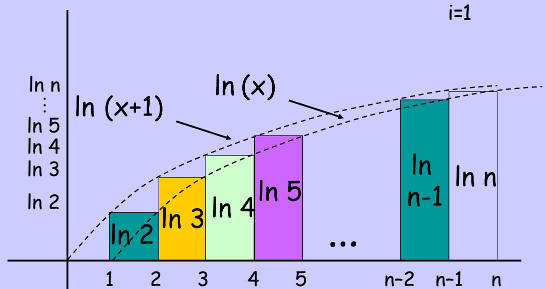
$$n! ::= 1 \times 2 \times 3 \times \cdots \times (n-1) \times n = \prod_{i=1}^{n} i$$

Turn product into a sum taking logs:

$$\ln(n!) = \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n)$$
$$= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n)$$
$$= \sum_{i=1}^{n} \ln(i)$$

Integral Method

Integral method to bound $\sum_{i=1}^{n} \ln i$



Integral Method on ln(n!)

$$\int_{1}^{n} \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_{0}^{n} \ln(x+1)dx$$

Integral Method on ln(n!)

Reminder:

$$\int \ln x\,dx = x \ln\left(\frac{x}{e}\right)$$

Integral Method on ln(n!)

$$\int_{1}^{n} \ln(x)\,dx \le \sum_{i=1}^{n} \ln(i) \le \int_{0}^{n} \ln(x+1)dx$$

$$n \ln(n/e) + 1 \le \sum \ln(i) \le (n+1) \ln((n+1)/e) + 1$$

so guess: $\sum_{i=1}^{n} \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$

Integral Method

$$\sum_{i=1}^{n} \ln(i) \approx \left(n+\frac{1}{2}\right)\ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$$

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Stirling's Formula

A tighter approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

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Asymptotic Equivalence

Def. $f(n) \sim g(n)$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

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Asymptotic Equivalence ~

Example: $(n^2+n) \sim n^2$

because

$$\lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim[\tfrac{n^2}{n^2} + \tfrac{n}{n^2}]$$
$$= \lim[1+\tfrac{1}{n}]$$
$$= 1 + \lim \tfrac{1}{n}$$
$$= 1 + 0 = 1$$

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Little Oh: $o(\cdot)$

Asymptotically smaller :

Def. $f(n) = o(g(n))$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

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Little Oh: $o(\cdot)$

$$n^2 = o(n^3)$$

because

$$\lim_{n\to\infty} \frac{n^2}{n^3} =$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

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Big Oh: O(·)

Asymptotic Order of Growth:

f(n) = O(g(n))

$$\lim\cancel{\sup}_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) < \infty$$

a technicality -- ignore now

Big Oh: O(·)

$$3n^2 = O(n^2)$$

because

$$\lim_{n\to\infty}\frac{3n^2}{n^2}$$

$$= 3 < \infty$$

The Oh's

Lemma:

If $f = o(g)$ or $f \sim g$, then $f = O(g)$

lim = 0 or lim = 1 ⟶ lim < ∞

The Oh's

If $f = o(g)$, then $g \neq O(f)$

$$\lim \frac{f}{g} = 0 \quad \longrightarrow \quad \lim \frac{g}{f} = \infty$$

Big Oh: O(·)

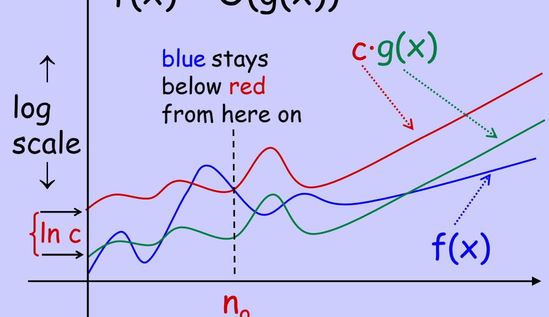
Equivalent definition:

f(n) = O(g(n))

$$\exists c, n_0 \geq 0\ \forall n \geq n_0.\ |f(n)| \leq c \cdot g(n)$$

Big Oh: O(·)

f(x) = O(g(x))



Team Problems

Problems 1&2

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Little Oh: $o(\cdot)$

Lemma: $\boxed{x^a = o(x^b) \text{ for } a < b}$

Proof: $$\frac{x^a}{x^b} = \frac{1}{x^{b-a}} \quad \text{and } b - a > 0$$

so as $x \to \infty$, $\frac{1}{x^{b-a}} \to 0$

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Little Oh: $o(\cdot)$

Lemma:

$$\boxed{\ln x = o(x^\delta)}$$

for $\delta > 0$.

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Little Oh: $o(\cdot)$

Lemma: $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $$\frac{1}{y} \le y \quad \text{for } y \ge 1$$

$$\int_1^z \frac{1}{y} dy \le \int_1^z y dy$$

$$\ln z \le \frac{z^2 - 1}{2}$$

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Little Oh: $o(\cdot)$

Lemma: $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $\ln z \le \frac{z^2}{2}$, so let $z ::= \sqrt{x^\varepsilon}$

$$\frac{\varepsilon \ln x}{2} \le \frac{x^\varepsilon}{2}$$

$$\ln x \le \frac{x^\varepsilon}{\varepsilon} = o(x^\delta) \quad \text{for } \delta > \varepsilon.$$

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Little Oh: $o(\cdot)$

Other proofs:
L'Hopital's Rule,
McLaurin Series
(see a Calculus text)

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Theta: $\Theta(\cdot)$

Same Order of Growth:

$$f(n) = \Theta(g(n))$$

$$f(n)=O(g(n)) \text{ and } g(n)=O(f(n))$$

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Big Oh Mistakes

f = O(g) defines a relation "= O(·)"

Don't write O(g) = f.

Otherwise: x = O(x), so O(x) = x.

But 2x = O(x), so

2x = O(x) = x,

therefore 2x = x.

Nonsense!

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Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

Of course really:

$$\sum_{i=1}^{n} i = \Theta(n^2)$$

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Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

false proof:

0 = O(1), 1 = O(1), 2 = O(1),...

So each i = O(1). So

$$\sum_{i=1}^{n} i = O(1) + O(1) + \cdots + O(1)$$

$$= n \cdot O(1) = O(n).$$

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Team Problems

Problems 3&4

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