



Mathematics for Computer Science
MIT 6.042J/18.062J

Sums & Money

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April 6, 2007

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C. F. Gauss



Picture source: <http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Gauss.html>

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Sum for Children

$$\begin{array}{rcl} 89 & + & 102 + 115 + 128 + 141 + \\ 154 & + & \dots + \\ 193 & + & \dots + \\ 232 & + & \dots + \\ 323 & + & \dots + \\ 414 & + & \dots + 453 + 466 \end{array}$$

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Sum for Children

Nine-year old Gauss saw
30 numbers, each 13 greater
than the previous one.
(So the story goes.)

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Sum for Children

$$\begin{array}{rcl} 1^{\text{st}} + 30^{\text{th}} & = & 89 + 466 = 555 \\ 2^{\text{nd}} + 29^{\text{th}} & = & \\ (1^{\text{st}}+13) + (30^{\text{th}}-13) & = & 555 \\ 3^{\text{rd}} + 28^{\text{th}} & = & \\ (2^{\text{nd}}+13) + (29^{\text{th}}-13) & = & 555 \\ & \vdots & \end{array}$$

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Sum for Children

Sum of k^{th} term and $(31-k)^{\text{th}}$ term
is **invariant!** 15 pairs of terms, so
Total = $555 \cdot 15$
 $= (1^{\text{st}} + \text{last}) \cdot (\# \text{ terms}/2)$
 $= \underbrace{(1^{\text{st}} + \text{last})/2}_{\text{average term}} \cdot (\# \text{ terms})$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Sum for Children

Example:

$$1 + 2 + \dots + (n-1) + n = \frac{(1+n)n}{2}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Geometric Series

$$G_n ::= 1 + x + x^2 + \dots + x^{n-1} + x^n$$

$$xG_n = x + x^2 + x^3 + \dots + x^n + x^{n+1}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Geometric Series

$$G_n ::= 1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^{n-1}} + \cancel{x^n}$$

$$xG_n = \cancel{x} + \cancel{x^2} + \cancel{x^3} + \dots + \cancel{x^n} + x^{n+1}$$

$$G_n - xG_n = 1 - x^{n+1}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Geometric Series

$$G_n ::= 1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^{n-1}} + \cancel{x^n}$$

$$xG_n = \cancel{x} + \cancel{x^2} + \cancel{x^3} + \dots + \cancel{x^n} + x^{n+1}$$

$$G_n - xG_n = 1 - x^{n+1}$$

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Consider *infinite* sum (series)

$$1 + x + x^2 + \dots + x^{n-1} + x^n + \dots = \sum_{i=0}^{\infty} x^i$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Infinite Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

$$\lim_{n \rightarrow \infty} G_n = \frac{1 - \lim_{n \rightarrow \infty} x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Infinite Geometric Series

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

for $|x| < 1$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Team Problem

Problem 1

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The future value of \$\$

I will pay you \$100 in 1 year,
if you will pay me \$X now.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The future value of \$\$

My bank will pay me 3% interest.
define *bankrate*

$b ::= 1.03$

-- bank increases my \$ by this
factor in 1 year.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The future value of \$\$

If I deposit your \$X now,
I will have $b \cdot X$ in 1 year.
So I won't lose money as long as

$$b \cdot X \geq 100.$$

$$X \geq \$100/1.03 \approx \$97.09$$

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12		10	5
3	1	4	14
15	8	11	2

The future value of \$\$

- \$1 in 1 year is worth \$0.9709 now.
- \$r last year is worth \$1 today,
where $r ::= 1/b$.
- So \$n paid in 2 years is worth
\$nr paid in 1 year, and is worth
\$nr² today.

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12		10	5
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The future value of \$\$

\$ n paid k years from now
is worth $\$n \cdot r^k$ today
where $r ::= 1/\text{bankrate}$.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Annuities

I pay you \$100/year for 10 years,
if you will pay me \$ y now.

I *can't lose* if you pay me

$$\begin{aligned} &100r + 100r^2 + 100r^3 + \dots + 100r^{10} \\ &= 100r(1 + r + \dots + r^9) \\ &= 100r(1 - r^{10})/(1 - r) = \$853.02 \end{aligned}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Annuities

I pay you \$100/year for 10 years,
if you will pay me \$853.02.

QUICKIE: If bankrates unexpectedly
increase in the next few years,

- A. You come out ahead
- B. The deal stays fair
- C. I come out ahead

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Manipulating Sums

$$\frac{d}{dx} \left(\sum_{i=0}^n x^i \right) = \frac{d}{dx} \left(\frac{1 - x^{n+1}}{1 - x} \right)$$

$$\sum_{i=0}^n i x^{i-1} = \frac{1}{x} \sum_{i=1}^n i x^i = \frac{d}{dx} \left(\frac{1 - x^{n+1}}{1 - x} \right)$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Manipulating Sums

$$\sum_{i=1}^n i x^{i-1} = \frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$$

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6	9	13	7
12		10	5
3	1	4	14
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Team Problems

Problems

2&3

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