Mathematics for Computer Science
MIT 6.042J/18.062J

Sums & Money

C. F. Gauss

Picture source: http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Gauss.html

Sum for Children

$$\begin{array}{l} 89 + 102 + 115 + 128 + 141 + \\ 154 + \cdots + \\ 193 + \cdots + \\ 232 + \cdots + \\ 323 + \cdots + \\ 414 + \cdots + 453 + 466 \end{array}$$

Sum for Children

Nine-year old Gauss saw

30 numbers, each 13 greater than the previous one.

(So the story goes.)

Sum for Children

$1^{st} + 30^{th} = 89 + 466 = 555$

$2^{nd} + 29^{th} =$
$(1^{st}+13) + (30^{th}-13) = 555$

$3^{rd} + 28^{th} =$
$(2^{nd}+13) + (29^{th}-13) = 555$

$\vdots$

Sum for Children

Sum of k^{th} term and $(31-k)^{th}$ term is invariant! 15 pairs of terms, so

$$\begin{aligned} \text{Total} &= 555 \cdot 15 \\ &= (1^{st} + \text{last}) \cdot (\#\text{ terms}/2) \\ &= \underbrace{(1^{st} + \text{last})/2}_{\text{average term}} \cdot (\#\text{ terms}) \end{aligned}$$

Sum for Children

Example:

$$1 + 2 + \cdots + (n-1) + n = \frac{(1+n)n}{2}$$

Geometric Series

$$G_n ::= 1 + x + x^2 + \cdots + x^{n-1} + x^n$$
$$xG_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$$

Geometric Series

$$G_n ::= 1 + \cancel{x} + \cancel{x^2} + \cdots + x^{n-1} + \cancel{x^n}$$
$$xG_n = \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots + \cancel{x^n} + x^{n+1}$$

$$G_n - xG_n = 1 \qquad - x^{n+1}$$

Geometric Series

$$G_n ::= 1 + \cancel{x} + \cancel{x^2} + \cdots + x^{n-1} + \cancel{x^n}$$
$$xG_n = \cancel{x} + \cancel{x^2} + \cancel{x^3} + \cdots + \cancel{x^n} + x^{n+1}$$

$$G_n - xG_n = 1 - x^{n+1}$$

$$\boxed{G_n = \frac{1 - x^{n+1}}{1 - x}}$$

Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

Consider *infinite* sum (series)

$$1 + x + x^2 + \cdots + x^{n-1} + x^n + \cdots = \sum_{i=0}^{\infty} x^i$$

Infinite Geometric Series

$$G_n = \frac{1 - x^{n+1}}{1 - x}$$

$$\lim_{n\to\infty} G_n = \frac{1 - \lim_{n\to\infty} x^{n+1}}{1 - x} = \frac{1}{1 - x}$$

Infinite Geometric Series

$$\boxed{\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}}$$

for $|x| < 1$

Team Problem

Problem 1

The future value of $$

I will pay you \$100 in 1 year, if you will pay me \$X now.

The future value of $$

My bank will pay me 3% interest.

define *bankrate*

$b ::= 1.03$

-- bank increases my \$ by this factor in 1 year.

The future value of $$

If I deposit your \$X now,
I will have \$b·X in 1 year.
So I won't lose money as long as

$$b \cdot X \geq 100.$$

$$X \geq \$100/1.03 \approx \$97.09$$

The future value of $$

- \$1 in 1 year is worth \$0.9709 now.
- \$r last year is worth \$1 today, where $r ::= 1/b$.
- So \$n paid in 2 years is worth \$nr paid in 1 year, and is worth \$$nr^2$ today.

The future value of $$

$n paid k years from now is worth $$n \cdot r^k$ today where $r ::= 1/\text{bankrate}$.

Annuities

I pay you $100/year for 10 years, if you will pay me $Y now.
I *can't lose* if you pay me

$$100r + 100r^2 + 100r^3 + \cdots + 100r^{10}$$
$$= 100r(1 + r + \cdots + r^9)$$
$$= 100r(1-r^{10})/(1-r) = \$853.02$$

Annuities

I pay you $100/year for 10 years, if you will pay me $853.02.
QUICKIE: If bankrates unexpectedly increase in the next few years,

A. You come out ahead
B. The deal stays fair
C. I come out ahead

Manipulating Sums

$$\frac{d}{dx}\left(\sum_{i=0}^{n} x^i\right) = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right)$$

$$\sum_{i=0}^{n} ix^{i-1} = \frac{1}{x}\sum_{i=1}^{n} ix^i = \frac{d}{dx}\left(\frac{1-x^{n+1}}{1-x}\right)$$

Manipulating Sums

$$\sum_{i=1}^{n} ix^{i-1} = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$$

Team Problems

Problems 2&3