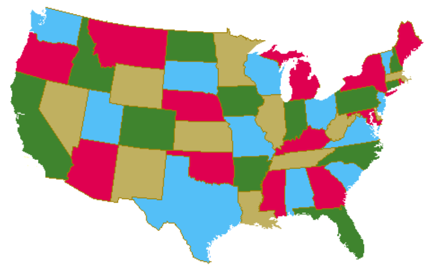
Mathematics for Computer Science
MIT 6.042J/18.062J

Planar Graphs

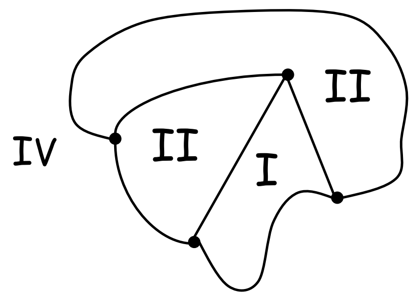
Planar Graphs



Planar Graphs

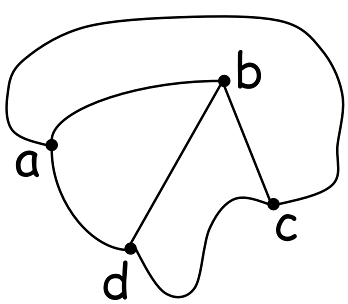
A graph is ***planar*** if there is a way to draw it in the plane without edges crossing.

Four Continuous Faces

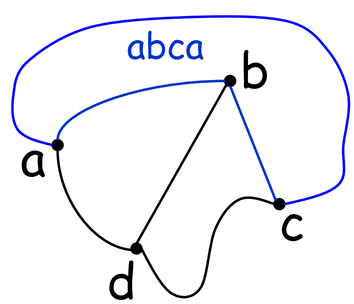


4 Connected Regions

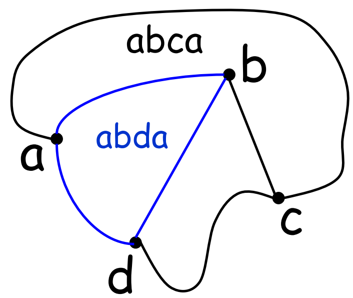
Region Boundaries



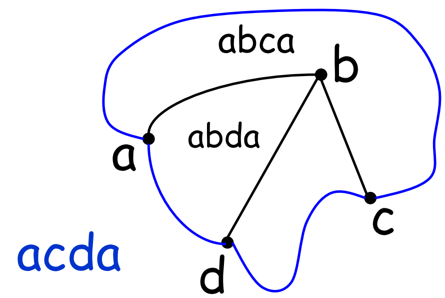
Region Boundaries



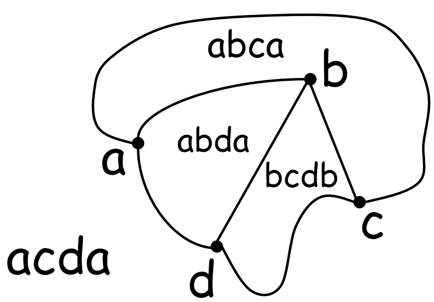
Region Boundaries



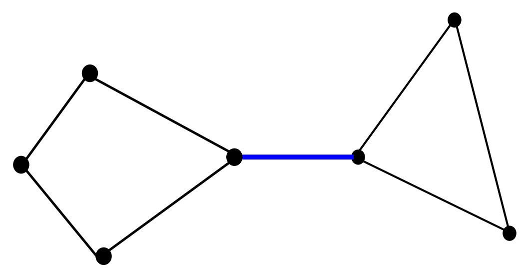
Region Boundaries



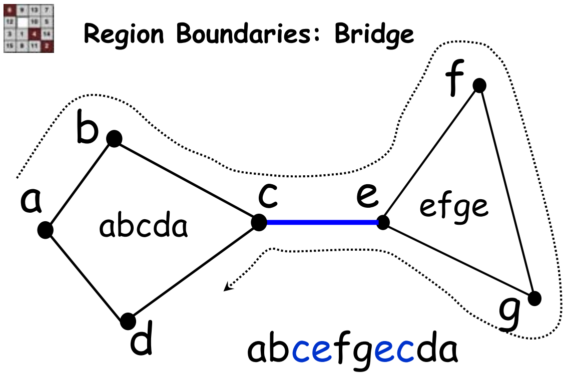
Region Boundaries



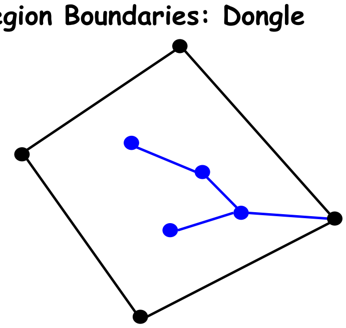
Region Boundaries: Bridge



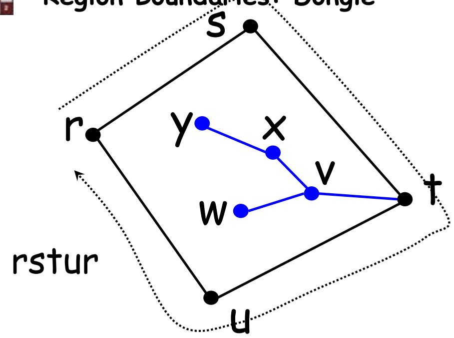
Region Boundaries: Bridge



Region Boundaries: Dongle



Region Boundaries: Dongle



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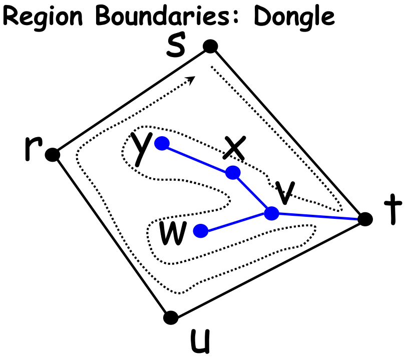
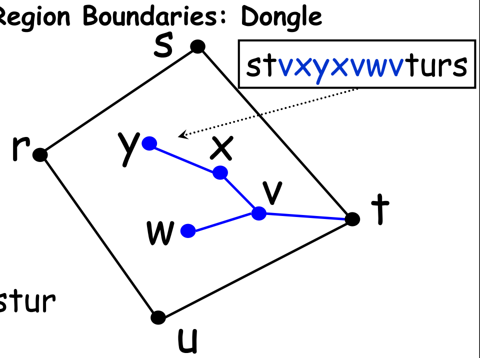
Region Boundaries: Dongle



Region Boundaries: Dongle



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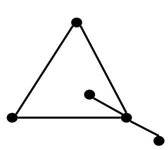
Planar Embedding

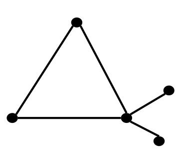
A **planar embedding** is a graph *along with* its face boundaries: cycles

(same graph may have different embeddings)

Same graph, different embeddings



length 5 faces



length 7 face

Recursive Def: Planar Embeddings

Base: a graph consisting of a single vertex, *v*,

along with face: length 0 cycle from *v* to *v*,

is a PE.

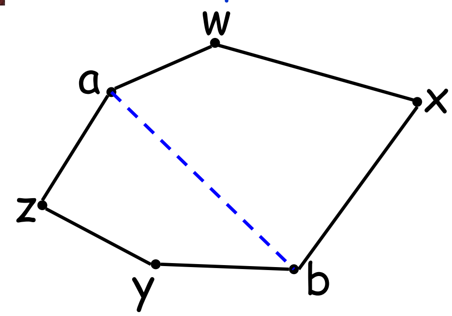
graph faces

Adding an edge to an embedding

Two constructor cases:

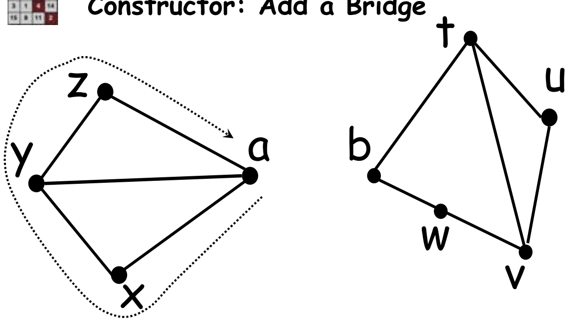
1) Add edge across a face (splits face in two)
2) Add bridge between components (merges 2 outer faces)

Constructor: Split a Face



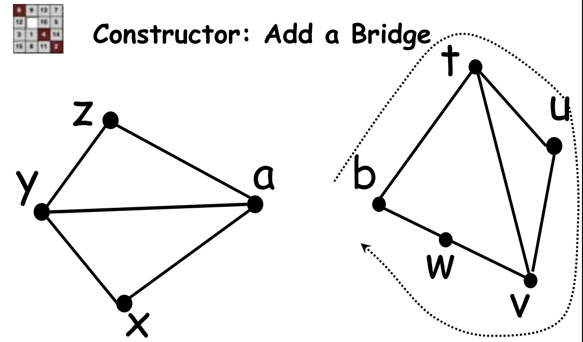
awxbyza → awxba, abyza

Constructor: Add a Bridge



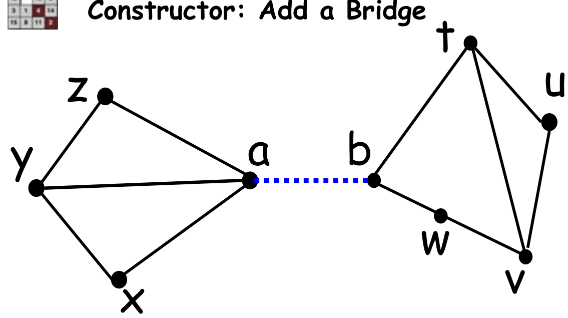
axyza

Constructor: Add a Bridge



axyza btuvwb

Constructor: Add a Bridge



axyza, btuvwb → axyzabtuvwba

Euler's Formula

If a planar embedding has v vertices, e edges, and f faces, then

$$v - e + f = 2$$

Euler's Formula

- Proof by structural induction on embeddings:
- **base case:** 1 vertex

$$v = 1,\ f = 1,\ e = 0$$
$$1 - 0 + 1 = 2$$

Adding an edge to a drawing

Constructor case (split face):

- v stays the same
- e increases by 1
- f increases by 1

so $v - e + f$ stays the same

Adding an edge to a drawing

Constructor case (add bridge):

- $v = v_1 + v_2$
- $e = e_1 + e_2 + 1$
- $f = f_1 + f_2 - 1$

$(v_1 + v_2) - (e_1 + e_2 + 1) + (f_1 + f_2 - 1)$

$= 2 + 2 - 2 = 2$

Euler's Formula

Corollary:

There are at most 5 regular polyhedra

(proof in Notes)

Planar Properties

- each edge appears twice on faces
- face length ≥ 3 (for $v \geq 3$)

so $3f \leq 2e$

combining with Euler:

$$\boxed{e \leq 3v - 6}$$

Planar Properties

- each edge appears twice on faces
- face length ≥ 3 (for $v \geq 3$)
- can draw edges in any order

(proofs by structural induction)

Planar Properties: Corollaries

- K_5 and $K_{3,3}$ not planar
- $\exists$ vertex of degree ≤ 5
- subgraphs are planar
- 6-colorable

 March 19, 2007 lec 7m.31

Team Problems

Problems 1–3

 March 19, 2007 lec 7m.32