

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Bipartite Matching

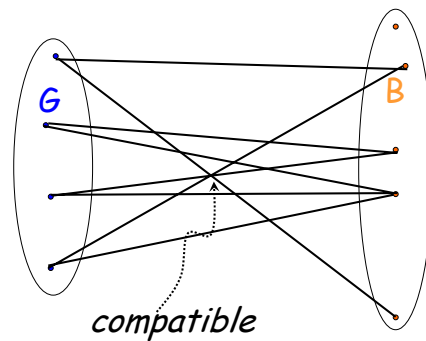
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March 14, 2007

lec 6W.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Compatible Boys & Girls



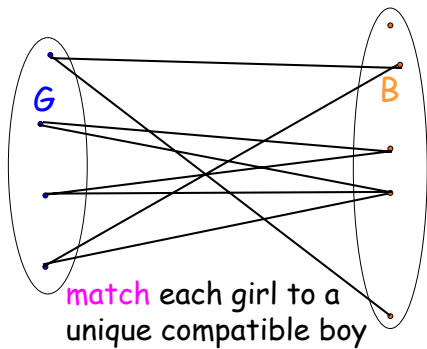
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lec 6W.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Compatible Boys & Girls



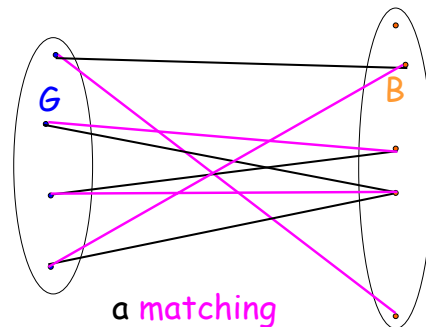
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lec 6W.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Compatible Boys & Girls



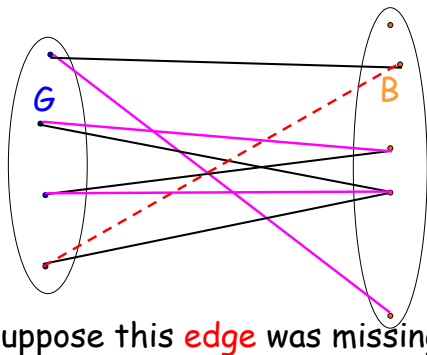
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lec 6W.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Compatible Boys & Girls



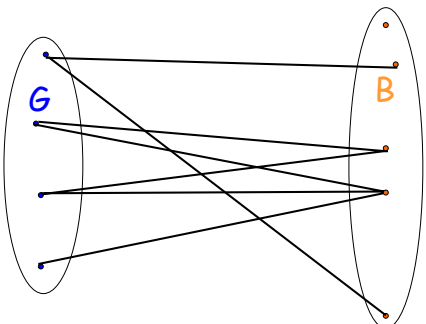
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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

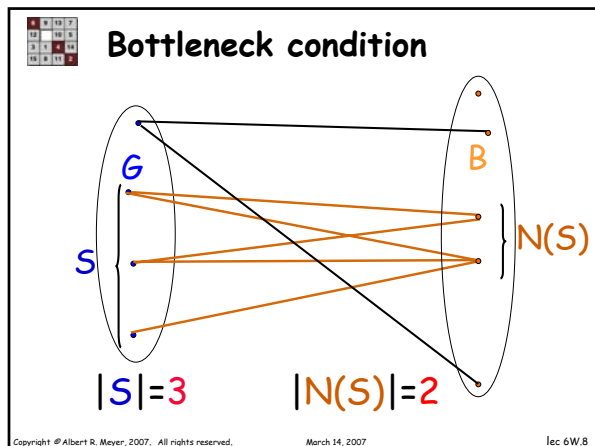
No match possible



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Bottleneck Lemma

bottleneck: not enough boys for some set of girls.

$N(S) ::= \{b \mid b \text{ adjacent to an } s \in S\},$
 $|S| > |N(S)|$

If there is a bottleneck,
 then no match is possible.

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Hall's Theorem

Conversely, if there are
 no bottlenecks, then
 there is a perfect match

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Hall's Theorem

Assume no bottlenecks.

Lemma: If S is a set of girls and
 $|S| = |N(S)|,$
 then there are no
 bottlenecks within S
 (obviously)

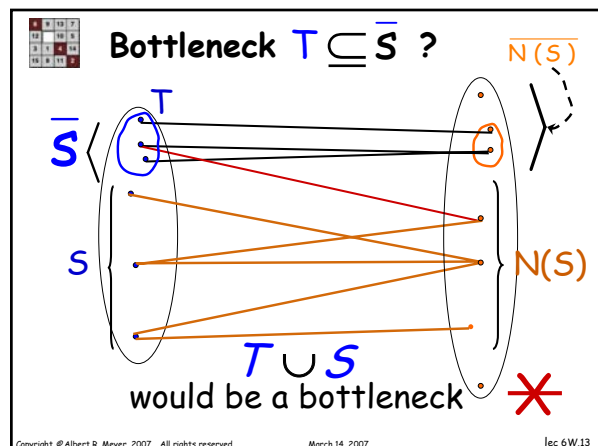
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Hall's Theorem

Assume no bottlenecks.

Lemma: If S is a set of girls and
 $|S| = |N(S)|,$
 and no bottlenecks
 between \bar{S} and $N(S)$

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Hall's Theorem

no bottlenecks implies
perfect match

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6	9	13	7
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Hall's Theorem

proof by induction on # girls.
case: proper subset, S ,
of girls with

$$|S| = |N(S)|$$

By Lemma no bottlenecks in
bipartite graph $(S, N(S))$,
and none in $(\overline{S}, \overline{N(S)})$

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6	9	13	7
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Hall's Theorem

by induction match
 $(S, N(S))$, and
 $(\overline{S}, \overline{N(S)})$
separately.

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6	9	13	7
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Hall's Theorem

case: $|S| < |N(S)|$ always.
match 1st girl with a boy.
remaining girls & boys won't
have any bottlenecks, so
by induction can match them

QED

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How to verify no bottlenecks?

Every girl likes $\geq d$ boys, and
every boy likes $\leq d$ girls,
implies no bottlenecks.

proof: any set S of girls with e
incident edges:

$$d|S| \leq e \leq d|N(S)|$$

$$|S| \leq |N(S)|$$

(no bottleneck)

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6	9	13	7
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Team Problem

Problems
1–3

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