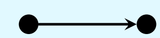
Mathematics for Computer Science
MIT 6.042J/18.062J

# Directed Graphs; Communication Networks

 March 16, 2007 

## Digraphs

a set, $V$, of *vertices*
a set, $E \subseteq V \times V$
of *directed* edges.
$(v,w) \in E$ notation: $v \to w$



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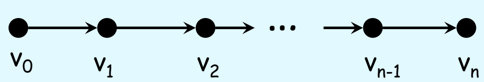
## Digraphs

paths are directed:

$v_0, v_1, \ldots, v_n$

where $v_i \to v_{i+1}$ for all i

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## Path Relation: Connectedness

v is *connected to* w:
there is a path

$v \to \cdots \to w$

(length 0 path from v to v)

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## Positive Path Relation

v is *connected to* w by a positive length path

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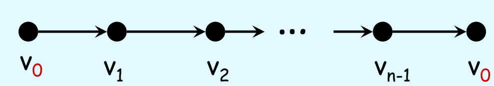
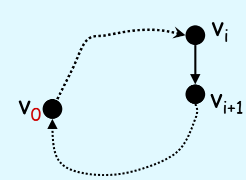
## Directed Cycles

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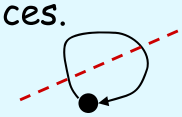
## Digraphs

Formally, a Digraph, D, is *exactly the same* as a binary relation on the vertices.

irreflexive:



asymmetric:

## Directed Acyclic Graph's

DAG's represent strict partial orders:

- The positive path relation of a DAG is a strict p.o.
- Every partial order is the positive path relation of a DAG.

## Communication Networks

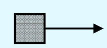
In particular,

*Permutation Networks*

## Permutation Networks

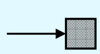
Digraphs with
n designated input vertices
with outdegree 1



## Permutation Networks

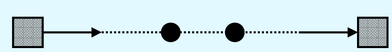
and with
n designated output vertices
with indegree 1



## Permutation Networks

and for *every* input and output, there is a path



switches

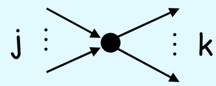
 

## Network Measures

diameter: largest input-output distance

size: # switches, # edges

switch degrees: $j \times k$

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## Permutation Routing Problems

A routing problem is a bijection,

$$\pi:\{1,\ldots n\} \to \{1,\ldots n\}$$

(called a *permutation*)

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## Permutation Routing Problems

A *solution* to a routing problem is a set of n paths from input k to input $\pi(k)$ for $k=1,\ldots,n$.

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## Permutation Problem Solutions

Solutions commonly select *shortest paths* between input k and output $\pi(k)$.

(but sometimes shortest paths are not best)

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## Permutation Problem Solutions

Quality of a *solution*:

latency: max path length

congestion: max #paths through one switch

(also average latency, congestion)

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## Difficulty of a Problem, $\pi$

*Problem* difficulty measured by *best* solution it allows:

problem-latency: *smallest* latency of any solution

problem-congestion: *smallest* congestion of any solution

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## Quality of A Network

Network quality measured by its *hardest* problem:

max-latency: *largest* problem-latency

max-congestion: *largest* problem-congestion

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## Quality of A Network

Finding max-congestion can be tricky. To prove max-con ≤ k: show how, given any problem, π, to route packets for π with congestion ≤ k.

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## Quality of A Network

Finding max-congestion can be tricky. To prove max-con ≥ k: must find problem, π, and show that every routing for π has congestion ≥ k.

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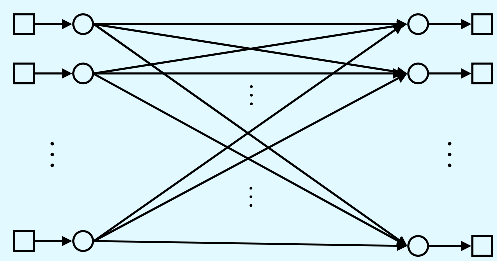
## A Good, Unreasonable Network



unique paths from in to out

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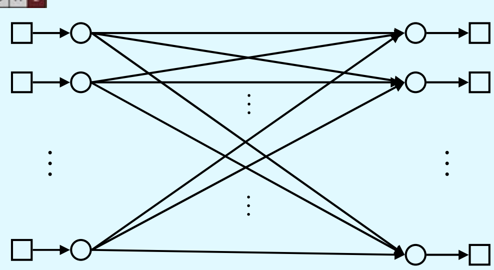
## A Good, Unreasonable Network



diameter = latency = 3

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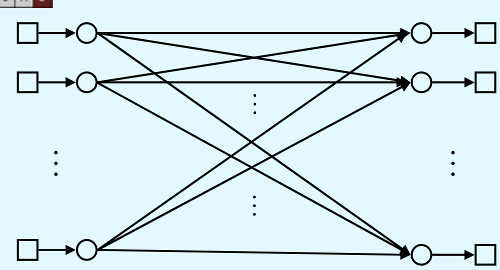
## A Good, Unreasonable Network



max-congestion = 1

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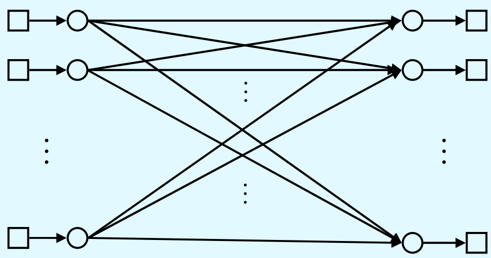
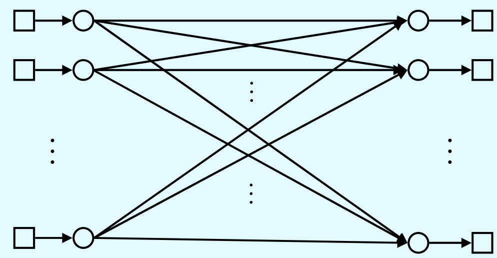
## A Good, Unreasonable Network



# switches = 2n



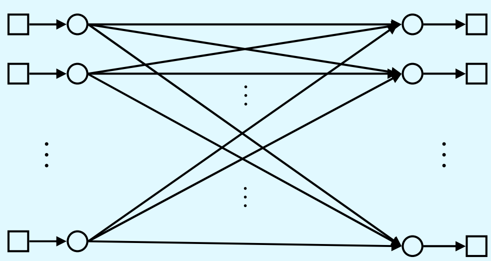
## A Good, Unreasonable Network



switch-degree: $1 \times n$, $n \times 1$



## A Good, Unreasonable Network



#edges: $n(n+2)$



## A Good, Unreasonable Network

Can be modified to use bounded switches (Class Problem 2).
Good in all ways
*but* $\approx n^2$ switches



# Benés Network



## A Great Network

Benés Network, $B_n$
handles

$$N ::= 2^n$$

inputs and outputs

## Benés Network

Benés Net is small:

latency $\approx$ 2 log N

#switches $\approx$ N log N

switch sizes = $1 \times 2$, $2 \times 1$

and max-congestion = 1

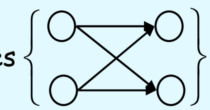
 

## Benés Network

Recursive Data Type

Base case: $B_1$

$2^1$ in-switches



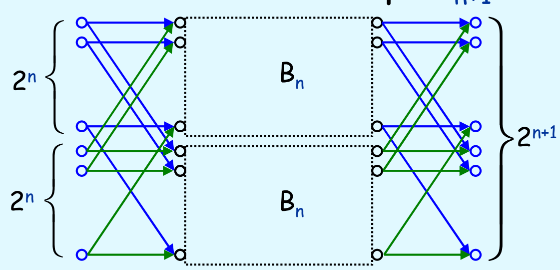
$2^1$ out-switches

N = 2

## Benés Network

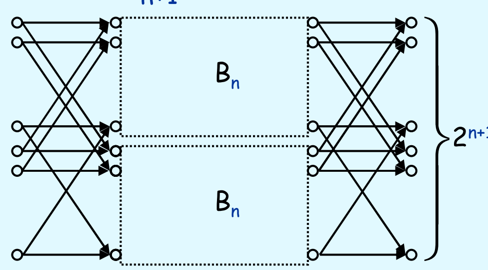
Constructor step: $B_{n+1}$



## Benés Network

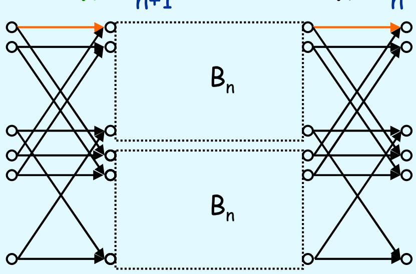
diam $B_{n+1}$



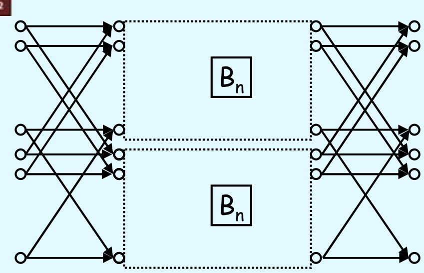
## Benés Network

diam $B_{n+1}$ = 2+ diam $B_n$



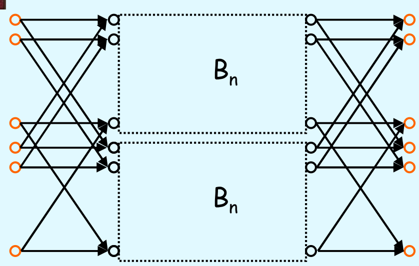
## Benés Network




size $B_{n+1}$ =

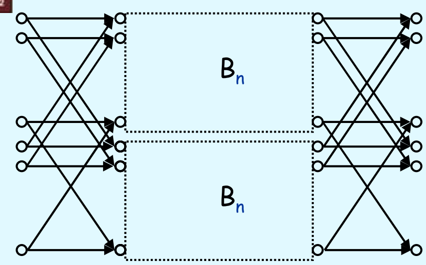
## Benés Network




size $B_{n+1}$ = 2 size $B_n$ + $2\cdot 2^{n+1}$

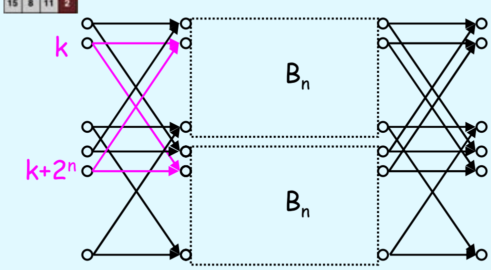
## Benés Network




for congestion 1:

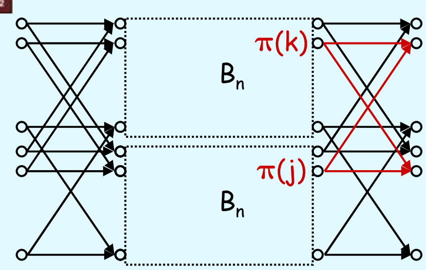
## Benés Network




for congestion 1: route to opposite halves

## Benés Network

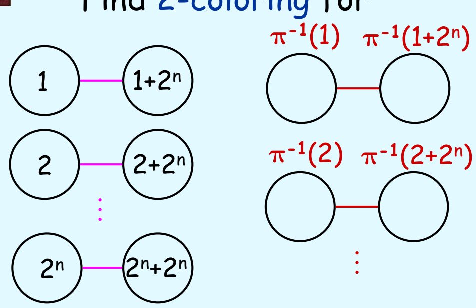



for congestion 1: route to opposite halves

## Benés Solution to $\pi$

Find 2-coloring for



## Team Problems

# Problems 1–3