



Mathematics for Computer Science

MIT 6.042J/18.062J

# Simple Graphs: Degrees, Isomorphism, Paths

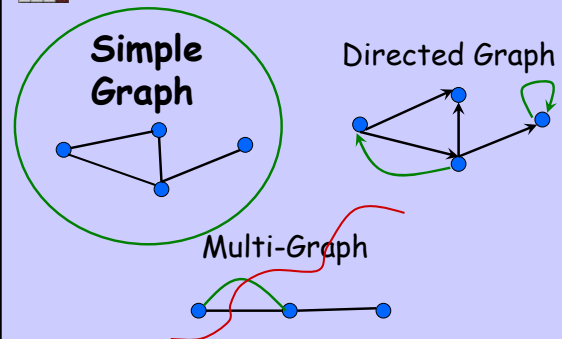
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March 7, 2007

lec 5W.1



## Types of Graphs



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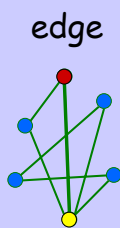
## A Simple Graph

vertices,  $V$

undirected edges,  $E$

$\{ \bullet, \bullet \}$

"adjacent"



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## Vertex Degree

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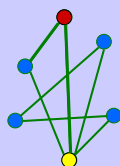
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## Vertex degree

*degree* of a vertex is  
# of *incident* edges

$\deg(\bullet) = 2$



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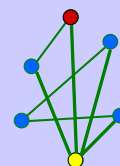
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## Vertex degree

*degree* of a vertex is  
# of *incident* edges

$\deg(\bullet) = 4$



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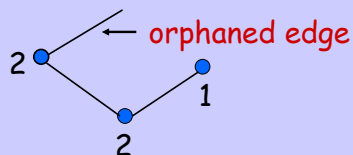
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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Impossible Graph

Is there a graph with  
vertex degrees 2,2,1?

NO!



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Handshaking Lemma

sum of degrees is  
twice # edges

$$2|E| = \sum_{v \in V} \deg(v)$$

$2+2+1 = \text{odd}$ ,  
so impossible

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Handshaking Lemma

sum of degrees is  
twice # edges

$$2|E| = \sum_{v \in V} \deg(v)$$

Proof: Each edge contributes  
2 to the sum on the right

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sex in America: Men more Promiscuous?

*Study claims:*

Men average many more  
partners than women.

Graph theory shows  
**this is nonsense**

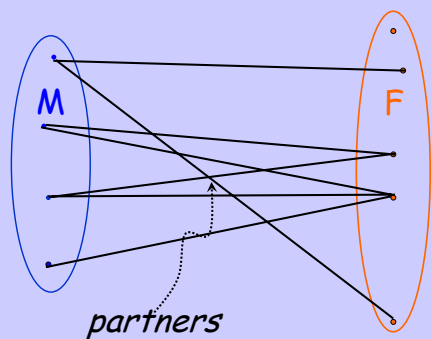
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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Sex Partner Graph



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Counting pairs of partners

$$\sum_{m \in M} \deg(m) = |E| = \sum_{f \in F} \deg(f)$$

divide both sides by  $|M|$

$$\underbrace{\frac{\sum_{m \in M} \deg(m)}{|M|}}_{\text{avg-deg}(M)} = \underbrace{\frac{\sum_{f \in F} \deg(f)}{|F|}}_{\text{avg-deg}(F)} \frac{|F|}{|M|}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Average number of partners

$$(\text{avg-deg}(M)) = (\text{avg-deg}(F)) 1.035$$

Averages differ solely by  
*ratio of females to males.*

No big difference  
Nothing to do with promiscuity.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

# Isomorphism

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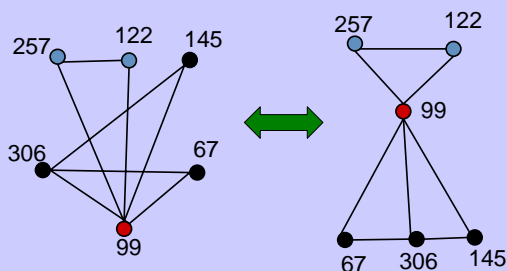
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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Graph Abstraction

Same graph (different *layouts*)



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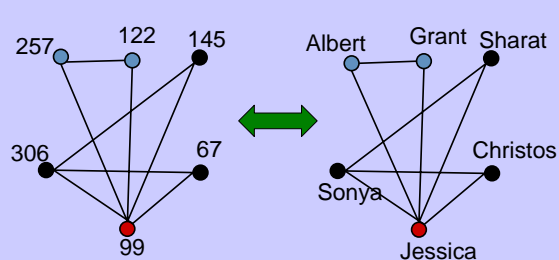
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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Graph Abstraction

Same graph (different *labels*)



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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Graph Abstraction

All that matters  
is the *connections*.  
Graphs with the  
same connections  
are *isomorphic*.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2


## Isomorphism

$G_1$  *isomorphic* to  $G_2$  means  
there is an *edge-preserving*  
*vertex matching*.

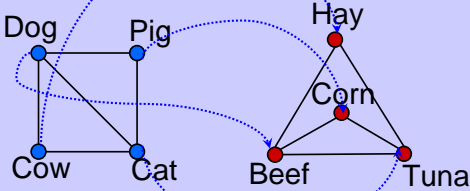
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### Are these Isomorphic?

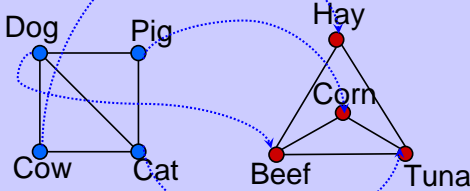


$f(\text{Dog}) = \text{Beef}$      $f(\text{Cow}) = \text{Hay}$   
 $f(\text{Cat}) = \text{Tuna}$      $f(\text{Pig}) = \text{Corn}$


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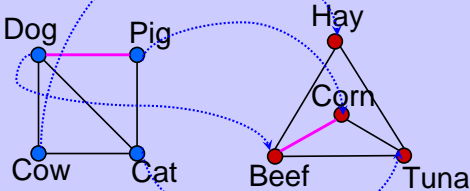
### Edges Preserved?




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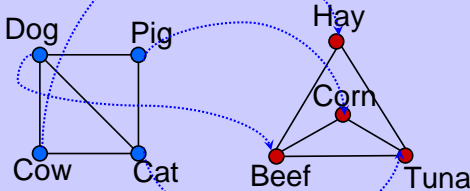
### Edges Preserved? YES!




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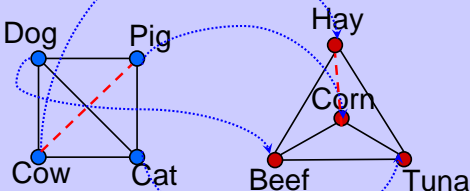
### NonEdges Preserved?




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### NonEdges Preserved? YES!



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### Graph Isomorphism

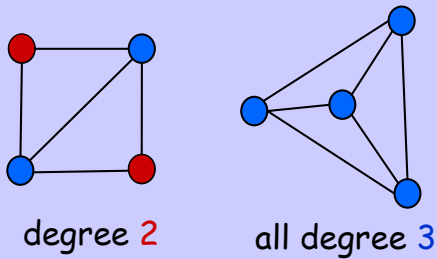
$G_1$  *isomorphic* to  $G_2$  means  
there is an *edge-preserving vertex matching*.

$\exists$  *bijection*  $f: V_1 \rightarrow V_2$   
 $u - v$  in  $E_1$  iff  $f(u) - f(v)$  in  $E_2$

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12		10	5
3	1	4	15
14	8	11	2

## Non-isomorphism



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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Finding the Mapping?

Not easy --many possible mappings.

Can test for *properties preserved under isomorphism*:

# of nodes, # edges,  
degree distributions,  
length of paths & cycles ...

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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Connectedness

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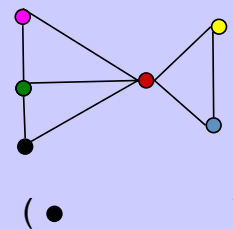
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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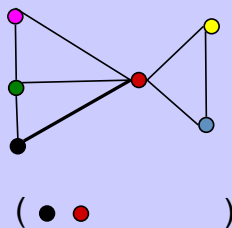
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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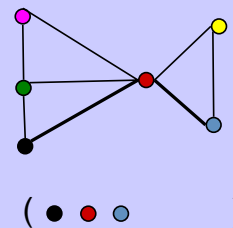
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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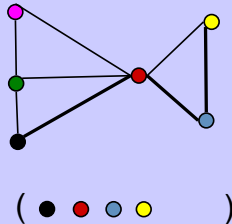
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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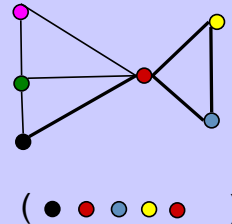
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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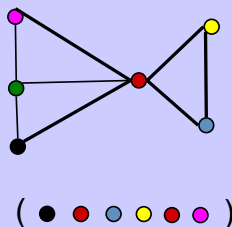
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Paths

*Path*: sequence of *adjacent* vertices



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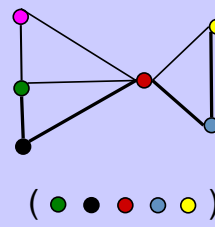
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Simple Paths

*Simple Path*: all vertices different



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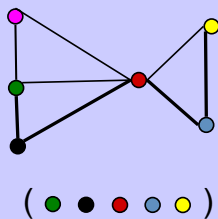
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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Simple Paths

*Simple Path*: (doesn't cross itself)



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6	9	13	7
12		10	5
3	1	4	15
14	8	11	2

## Connectedness

vertices  $v$ ,  $w$  are *connected* iff there is a path starting at  $v$  and ending at  $w$ .

A graph is *connected* iff every pair of vertices are connected.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Team Problems

# Problems 2,3,1,4

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