

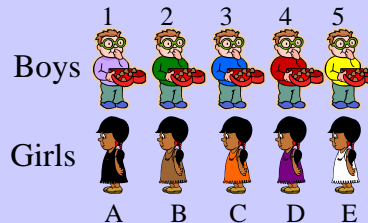


# Stable Matching



## Stable Marriage

### A Marriage Problem



## Stable Marriage

### Preferences:

Boys	Girls
1: CBEAD	A: 35214
2: ABECD	B: 52143
3: DCBAE	C: 43512
4: ACDBE	D: 12345
5: ABDEC	E: 23415



## Stable Marriage

### Preferences

1: CBEAD
2: ABECD
3: DCBAE
4: ACDBE
5: ABDEC

Try “greedy”  
strategy for boys



## Stable Marriage

### Preferences

1: <del>C</del> BEAD
2: ABE <del>C</del> D
3: D <del>C</del> BAE
4: A <del>C</del> DBE
5: ABDEC <del>C</del>

Marry Boy 1 with Girl C  
(his 1<sup>st</sup> choice)




## Stable Marriage

### Preferences

2: ABE D
3: D BAE
4: A DBE
5: ABDE

Marry Boy 1 with Girl C  
(his 1<sup>st</sup> choice)







### Stable Marriage

Preferences


Marry **Boy 1** with **Girl C**  
(his 1<sup>st</sup> choice)



2 : ABED  
3 : DBAE  
4 : ADBE  
5 : ABDE




1



C


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### Stable Marriage

Next:


Marry **Boy 2** with **Girl A**:  
(best remaining choice)



1 :  C  
~~2 : A~~ BED  
3 : DBAE  
4 : ~~A~~ DBE  
5 : ~~A~~ BDE




2




A

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### Stable Marriage


Final “boy greedy” marriages




1 C




2 A



3 D




4 B




5 E

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### Stable Marriage


**Trouble!**




1



C




4




B

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### Stable Marriage


Boy 4 **likes** Girl C **better** than his wife.




1



C




4




B

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### Stable Marriage


**and vice-versa**




1



C




4



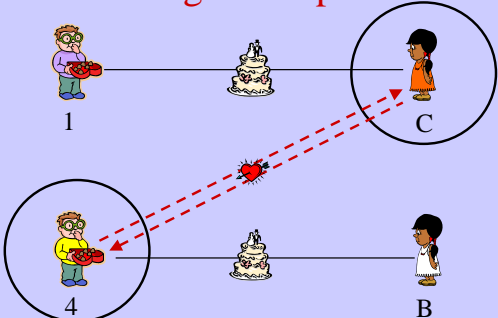
B

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### Stable Marriage


**Rogue Couple**



1 C

4 B

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### Stable Marriage

**Stable Marriage Problem:**

Marry everyone without any rogue couples!

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### Stable Marriage

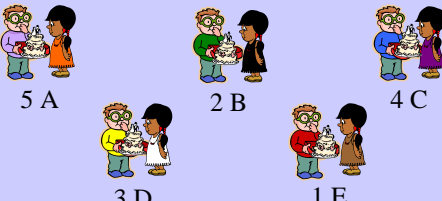
Let's Try it!

Unnumbered Class Problem

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### Stable Marriage I.




5 A 2 B 4 C

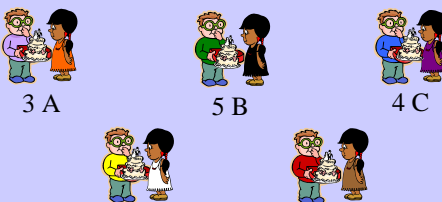
3 D 1 E

Boy Optimal

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### Stable Marriage II.




3 A 5 B 4 C

1 D 2 E

All Girls get 1st Choice

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### Stable Marriage

More than a puzzle:

- College Admissions  
(original Gale & Shapley paper, 1962)
- Matching Hospitals & Residents.
- Matching Dance Partners.

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## Stable Marriage



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## Stable Marriage

The Mating Ritual:  
*day by day*

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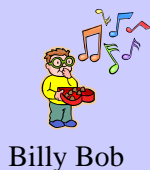
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## Mating Ritual

**Morning:** boy serenades favorite girl



Billy Bob



Brad



Angelina

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## Mating Ritual

**Morning:** boy serenades favorite girl

**Afternoon:** girl **rejects** all but favorite



Billy Bob



Brad



Angelina

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## Mating Ritual

**Morning:** boy serenades favorite girl

**Afternoon:** girl rejects all but favorite

**Evening:** rejected boy writes off girl



Billy Bob



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## Mating Ritual

**Stop** when no girl rejects.

Each girl marries her favorite  
suitor (if any).

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Ritual

### Partial Correctness:

- Everyone is married.
- Marriages are stable.

### Termination:

there exists a Wedding Day.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Stable Marriage: Termination

*total-boy's-list-length:*  
strictly decreasing &  $\mathbb{N}$ -valued.

So  $\exists$  Wedding Day.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Ritual: Girls improve

**Lemma:** A girl's favorite tomorrow will be at least as desirable as today's.

...because today's favorite will stay until she rejects him for someone better.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Ritual: Girls improve

**Lemma:** A girl's favorite tomorrow will be at least as desirable as today's.

(*favorite*( $G$ ) is **weakly increasing** for each  $G$ )

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Ritual: Boys Get Worse

**Lemma:** A boy's 1st love tomorrow will be no more desirable than today's.

...because boys work straight down their lists.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Algorithm: Boys Get Worse

**Lemma:** A boy's 1st love tomorrow will be no more desirable than today's.

(*serenading*( $B$ ) is **weakly decreasing** for each  $B$ )

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Mating Ritual: **Invariant**

If  $G$  is not on  $B$ 's list, then she has a better current favorite.

*Proof:* When  $G$  rejected  $B$  she had a better suitor, and  $favorite(G)$  is weakly increasing.

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Stable Marriage: Termination

On Wedding Day:

- Each girl has  $\leq 1$  suitors  
(by def of wedding day)
- Each boy is married, or has no girls on his list

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Mating Ritual: Everyone Marries

Everyone is Married by Wedding Day

*Proof:* by **contradiction**.

If  $B$  is not married, his list is empty.

By **Invariant**, all girls have favorites better than  $B$  -- so they *do* have a favorite.

That is, all **girls** are married, so all **boys** are married.

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Mating Ritual: Stable Marriages

Marriages are **Stable**:

Bob won't be in rogue couple with

case 1: a girl  $G$  **on** his final list, since **he's already married to the best of them**.

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Mating Ritual: Stable Marriages

Marriages are **Stable**:

Bob won't be in rogue couple with

case 2: a girl  $G$  **not on** his final list, since **by Invariant**,  $G$  likes her spouse better.

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4	8	13	7
12		10	5
3	1	6	14
15	9	11	2

### Mating Ritual

Who does better, boys or girls?

Girls' suitors get better, and boy's sweethearts get worse, so girls do better?

**No!**

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Mating Ritual

Mating Ritual is *Optimal* for all Boys at once. *Pessimal* for all Girls.

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Stable Marriage

More questions, rich theory:

Other stable marriages possible?

- Can be many.

Can a boy do better by lying? – No!

Can a girl do better by lying? – Yes!

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1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

## Team Problems

# Problems

# 1–3

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