Mathematics for Computer Science

MIT 6.042J/18.062J

State Machines, I: Invariants

 lec 4W.1

State machines

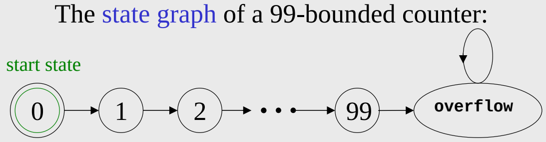
State machine:

Step by step procedure, possibly responding to input.

 lec 4W.2

State machines

The state graph of a 99-bounded counter:



States: $\{0,1,\ldots,99,$ `overflow`$\}$

Transitions: i → i+1 $\quad 0 \le i < 99$

99 → `overflow`

`overflow` → `overflow`

 lec 4W.3

Die Hard

Picture source: http://movieweb.com/movie/diehard3/

 lec 4W.4

Die Hard

Simon says: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

 lec 4W.5

Die Hard

Supplies:

Water

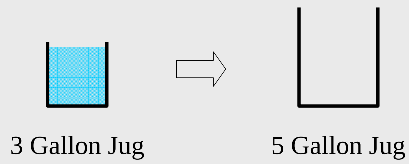
3 Gallon Jug

5 Gallon Jug

 lec 4W.6

Die Hard

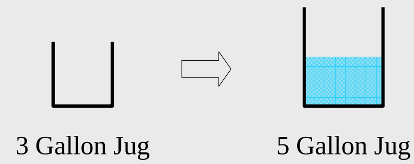
Transferring water:



 February 28, 2007 lec 4W.7

Die Hard

Transferring water:



 February 28, 2007 lec 4W.8

Die Hard

Simon's challenge:
Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will blow up.

Question: How to do it?

 February 28, 2007 lec 4W.9

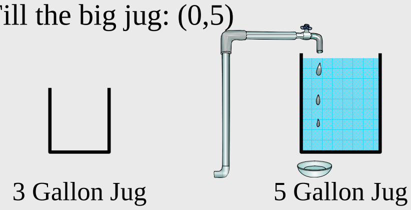
Die Hard

Work it out now!

 February 28, 2007 lec 4W.10

How to do it

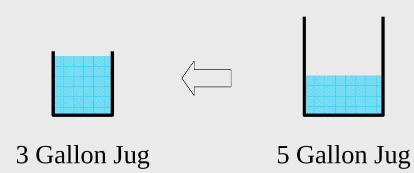
Start with empty jugs: (0,0)
Fill the big jug: (0,5)



 February 28, 2007 lec 4W.11

How to do it

Pour from big to little: (3,2)



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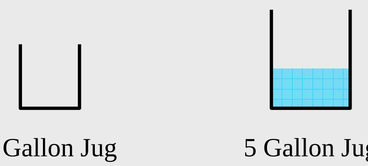
How to do it

Empty the little: (0,2)



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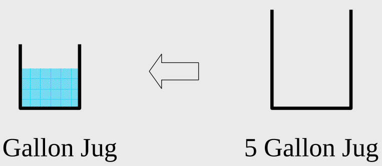
How to do it

Pour from big to little: (2,0)



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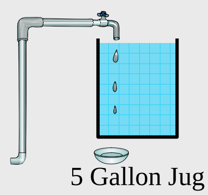
How to do it

Fill the big jug: (2,5)



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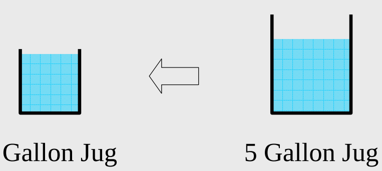
How to do it

Pour from big to little: (3,4)



Done!!

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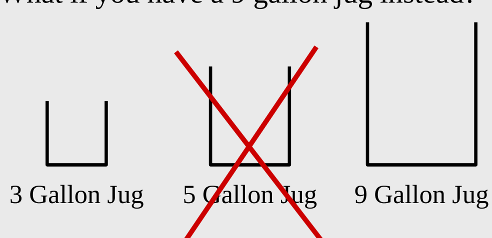
Die Hard once and for all

What if you have a 9 gallon jug instead?



Can you do it? Can you prove it?

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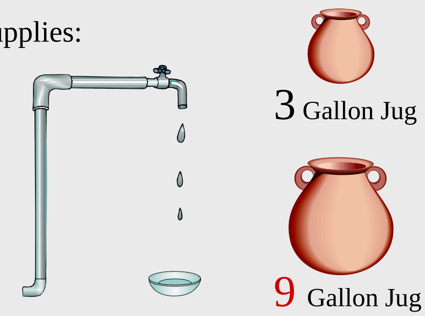
Die Hard

Work it out now!

 February 28, 2007 lec 4W.18

Die Hard Once & For All

Supplies:



3 Gallon Jug

9 Gallon Jug

Water

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State machines

Die hard state machine

State = amount of water in the jug: (b,l)

where $0 \le b \le 9$ and $0 \le l \le 3$.

Start State = (0,0)

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State machines

Die Hard Transitions:

1. Fill the little jug: $(b,l) \to (b,3)$ *for* $l < 3$
2. Fill the big jug: $(b,l) \to (9,l)$ *for* $b < 9$
3. Empty the little jug: $(b,l) \to (b,0)$ *for* $l > 0$
4. Empty the big jug: $(b,l) \to (0,l)$ *for* $b > 0$

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State machines

5. Pour from big jug into little jug (for $b > 0$):

(i) If $\underbrace{\text{no overflow}}_{b+l \leqq 3}$, then $(b,l) \to (0, b+l)$,

(ii) otherwise $(b,l) \to (b-(3-l), 3)$.

6. Pour from little jug into big jug.

Likewise.

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State Invariants

Die hard once and for all

Invariant:

P(state) ::= “3 divides the number of gallons in each jug.”

$$P((b,l)) ::= (3 \mid b \wedge 3 \mid l)$$

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State Invariants

Floyd’s Invariant Method

(just like induction)

1) Base case: Show $P(start)$.
2) Invariant case: Show

if $P(q)$ and $q \longrightarrow r$, then $P(r)$.

3) Conclusion: P holds for *all reachable states*, including final state (if any).

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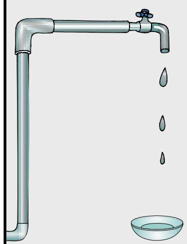
Die Hard Once & For All



Corollary: No state (4,x) is reachable, so Bruce Dies!

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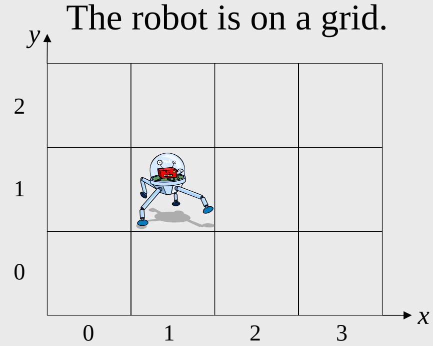
The Diagonal Robot

The robot is on a grid.



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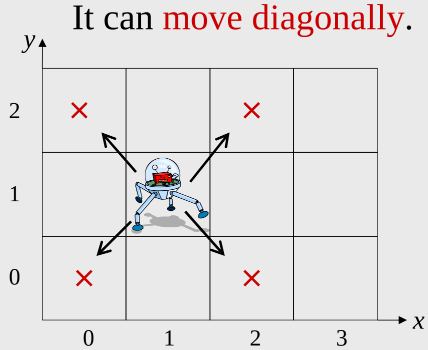
The Diagonal Robot

It can move diagonally.



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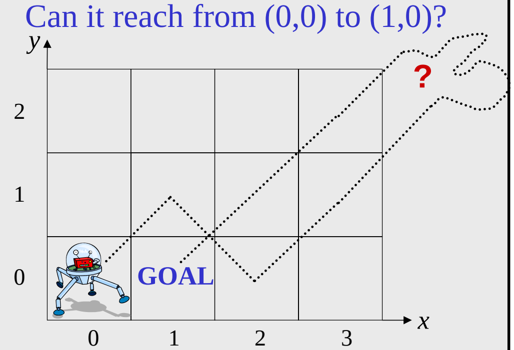
The Diagonal Robot

Can it reach from (0,0) to (1,0)?



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Robot Invariant

NO!

$P((x, y))$ ::= $x + y$ is even
is an invariant:
transition adds ±1 to **both** x and y,
preserving parity of $x+y$.
Also, $P((0, 0))$ is true.

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Robot Invariant

So all positions (x, y) reachable by robot have $x + y$ even.

But $1 + 0 = 1$ is odd, so (1,0) is not reachable.

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The Fifteen Puzzle Explained!

--by similar reasoning
(details in Team Problem 1)

Team Problems

Problems carried over to Friday