Induction

Example of Induction

Suppose we have a property (say *color*) of the nonnegative integers:

0, 1, 2, 3, 4, 5, …

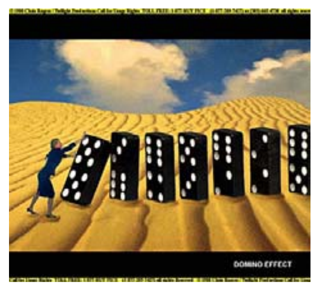
If *0* is *red*, and a number *next to* a red number is *red*, then *all numbers are red*!

The Induction Rule

0 and (from n to $n+1$), proves 0, 1, 2, 3,….

$$\frac{R(0), \quad \forall n \in \mathbb{N}. R(n) \rightarrow R(n+1)}{\forall m \in \underline{\mathbb{N}.}\ R(m)}$$

Like Dominos…



Example Induction Proof

Let's prove:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Proof by Induction

Statements in green form a template for inductive proofs.

- Proof: (by induction on n)
- The induction hypothesis, $P(n)$, is:

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Example Induction Proof

Base Case ($n = 0$):

$$\underbrace{1 + r + r^2 + \cdots + r^0}_{1} \overset{?}{=} \frac{r^{0+1} - 1}{r - 1}$$

$$= \frac{r-1}{r-1} = 1$$

Wait: divide by zero bug!
This is only true for $r \neq 1$

Correction

Theorem:

$$\forall r \neq 1. \ 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Induction Hypothesis:

$$P(n) ::= \forall r \neq 1. \ 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

An Example Proof

- Induction Step: Assume $P(n)$ for some $n \geq 0$ and prove $P(n+1)$:

$$\forall r \neq 1. \ 1 + r + r^2 + \cdots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

An Example Proof

Have $P(n)$ by assumption:

$$\forall r \neq 1. \ 1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

So let $r \in \mathbb{C}$ be any number $\neq 1$.

Then from $P(n)$ we have

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

An Example Proof

adding r^{n+1} to both sides,

$$1 + \cdots + r^n + r^{n+1} = \frac{r^{n+1} - 1}{r - 1} + r^{n+1}$$

$$= \frac{r^{n+1} - 1 + r^{n+1}(r - 1)}{r - 1}$$

$$= \frac{r^{(n+1)+1} - 1}{r - 1}$$

An Example Proof

That is,

$$1 + r + r^2 + \cdots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

But since $r \neq 1$ was arbitrary, we conclude (by UG), that

$$\forall r \neq 1. \ 1 + r + r^2 + \cdots + r^{n+1} = \frac{r^{(n+1)+1} - 1}{r - 1}$$

which is $P(n+1)$.

- This completes the induction proof. QED.

An Aside: Ellipsis

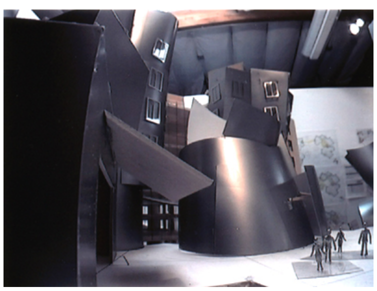
"…" is an *ellipsis*. It means the reader is supposed to *infer* a pattern:

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i$$

- Can lead to confusion (n = 0?)
- Summation notation notation more precise

The MIT Stata Center

The Stata Center Plaza



The Gehry/Gates Plaza

Goal: tile the squares, except one in the middle for Bill.

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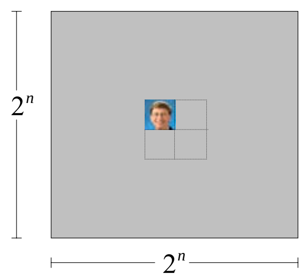
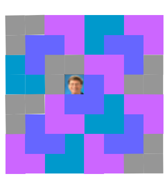


The Gehry/Gates Plaza

Gehry specifies L-shaped tiles covering three squares:

For example, for 8 x 8 plaza might tile for Bill this way:



The Gehry/Gates Plaza

Theorem: For any 2^n ▯ 2^n plaza, we can make Bill and Frank happy.

Proof: (by induction on n)

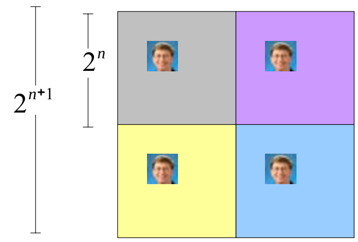
$P(n)$ ::= can tile 2^n ▯ 2^n with Bill in middle.

Base case: (n=0)

 (no tiles needed)

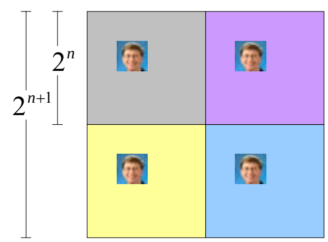
The Gehry/Gates Plaza

Induction step: assume can tile $2^n \times 2^n$, prove can handle $2^{n+1} \times 2^{n+1}$.



The Gehry/Gates Plaza

Now what?



The Gehry/Gates Plaza

The fix:

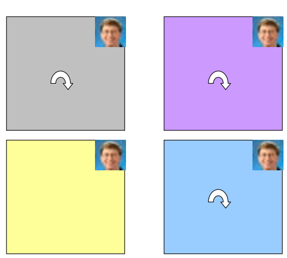
Prove that we can always find a tiling with Bill in the corner.

The Gehry/Gates Plaza

Note: Once have Bill in corner, can get Bill in middle:

The Gehry/Gates Plaza

Method:
Rotate the squares as indicated.

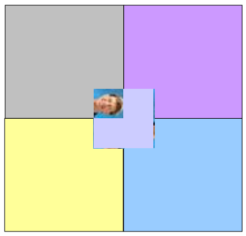


The Gehry/Gates Plaza

Method: after rotation have:

The Gehry/Gates Plaza

Method: Now group the 4 squares together, and insert a tile.



Done!
Bill in middle

The Gehry/Gates Plaza

Theorem: For any $2^n \times 2^n$ plaza, we can put Bill in the corner.

Proof: (by induction on n)

$P(n)$ ::= can tile $2^n \times 2^n$ with Bill in corner

Base case: (n=0)

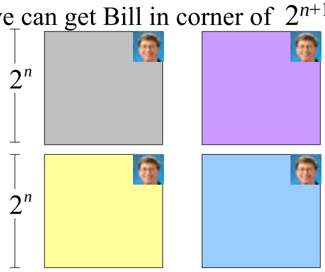
 (no tiles needed)

The Gehry/Gates Plaza

Induction step:

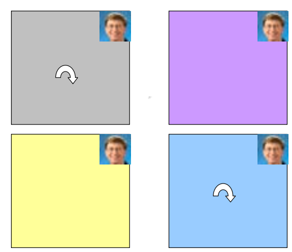
Assume we can get Bill in corner of $2^n \times 2^n$.

Prove we can get Bill in corner of $2^{n+1} \times 2^{n+1}$.



The Gehry/Gates Plaza

Method: Rotate the squares as indicated.

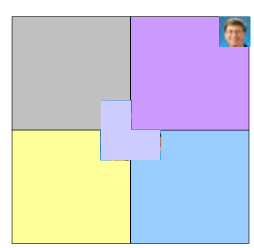


The Gehry/Gates Plaza

Method: Rotate the squares as indicated. after rotation have:

The Gehry/Gates Plaza

Method: Now group the squares together, and fill the center with a tile.



Done!

Ingenious Induction Hypotheses

Note 1: To prove "Bill in middle," we
proved *something else:*
"Bill in corner."

Ingenious Induction Hypotheses

Note 2: It may help to choose a *stronger hypothesis* than the desired result (class problem).

Recursive Procedure

Note 3: The induction proof of "Bill in corner" implicitly defines a recursive procedure for finding corner tilings.

A False Proof

Theorem: All horses are the same color.

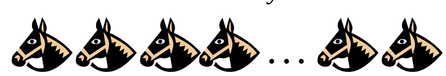
Proof: (by induction on n)

Induction hypothesis:

$P(n)$ ::= any set of n horses have the same color

Base case (n=0):

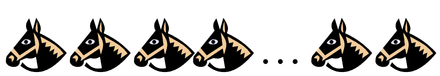
No horses so *vacuously* true!



A False Proof

(Inductive case)

Assume any n horses have the same color.
Prove that any $n+1$ horses have the same color.



$n+1$

A False Proof

(Inductive case)

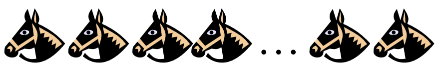
Assume any n horses have the same color.
Prove that any $n+1$ horses have the same color.

Second set of n horses have the same color

First set of n horses have the same color

A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.



Therefore the set of n+1 have the same color!

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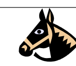
A False Proof

What is wrong? $n=1$

Proof that $P(n) \rightarrow P(n+1)$
is false if $n = 1$, because the two horse groups *do not overlap*.

Second set of n=1 horses



First set of n=1 horses

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A False Proof

Proof that $P(n) \rightarrow P(n+1)$
is false if $n = 1$, because the two horse groups *do not overlap*.

(But proof works for all $n \neq 1$)

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Team Problems

Problems 1–3

 February 21, 2007 lec 3w.40