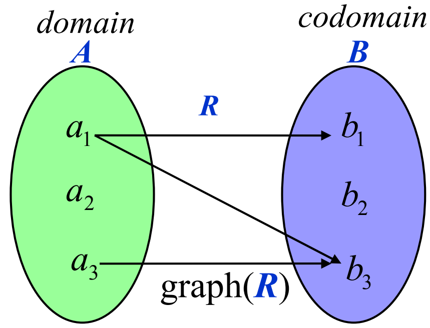
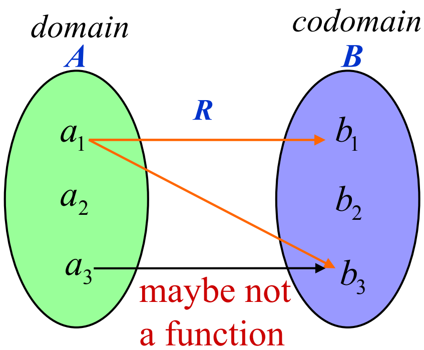
Mathematics for Computer Science
MIT 6.042J/18.062J

Partial Orders & Scheduling

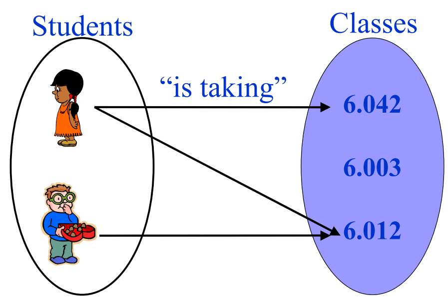
Binary relation *R* from *A* to *B*



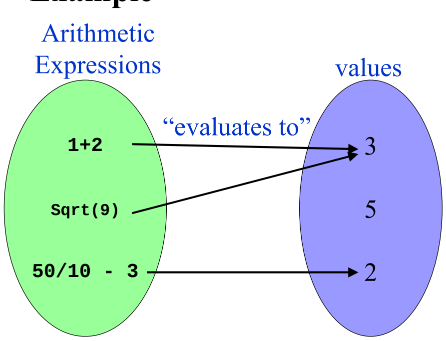
Binary relation *R* from *A* to *B*



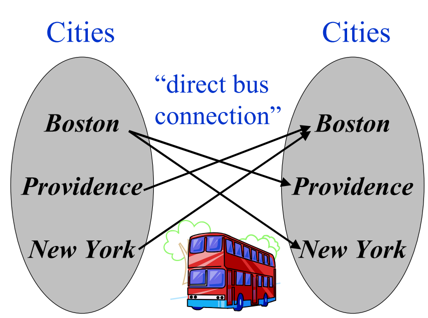
Example



Example



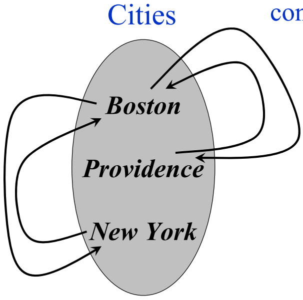
Example



Example

Cities

"direct bus connection"



Some Course 6 Prerequisites

18.01 → 6.042
18.01 → 18.02
18.01 → 18.03
8.01 → 8.02
6.001 → 6.034
6.042 → 6.046

18.03, 8.02 → 6.002
6.001, 6.002 → 6.004
6.001, 6.002 → 6.003
6.004 → 6.033
6.033 → 6.857
6.046 → 6.840

Subject Prerequisites

subject c is a direct prerequisite for subject d

$$c \longrightarrow d$$

Direct Prerequisites

18.01 ⟶ 6.042 ⟶ 6.046 ⟶ 6.840

Indirect Prerequisites

18.01 ⟶ 6.042 ⟶ 6.046 ⟶ 6.840

18.01 is *indirect* prereq. of 6.840

(⟶ is *transitive closure* of ⟶)

"Freshman subjects"

18.01 8.01 6.001

subjects with no prereqs:

d is a Freshman subject iff

$$\langle nothing \rangle \longrightarrow d$$

d is minim***al***

minimal not minimum

minimum means "smallest"

-- a prereq. for *every* subject

no minimum in this example

Constructing a Term Schedule

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

identify minim*al* elements

Constructing a Term Schedule

18.01 8.01 6.001

start schedule with them

Constructing a Term Schedule

- ~~18.01 → 6.042~~
- 18.01 → 18.02
- 18.01 → 18.03
- ~~8.01 → 8.02~~
- ~~6.001 → 6.034~~
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- ~~6.001~~, 6.002 → 6.004
- ~~6.001~~, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

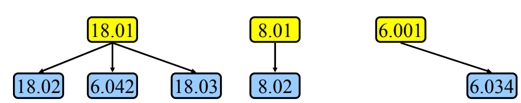
remove minimal elements

Constructing a Term Schedule

- 6.042
- 18.02
- 18.03
- 8.02
- 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.002 → 6.004
- 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

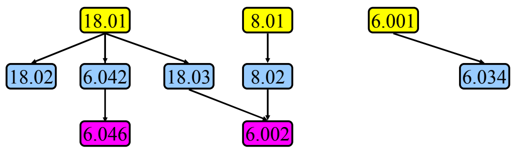
identify new minimal elements

Constructing a Term Schedule



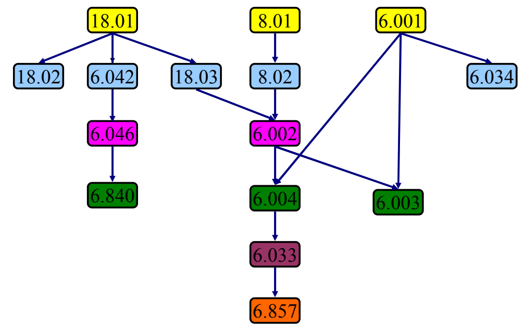
schedule them next

Constructing a Term Schedule



continue in this way…

Complete Term Schedule



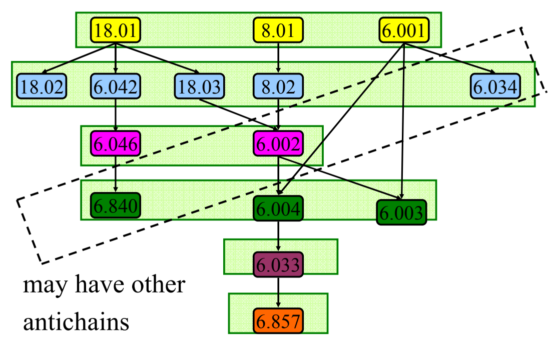
Antichains

Set of subjects with no prereqs among them

-- can be taken in *any order*.

(said to be *incomparable*)

Some Antichains



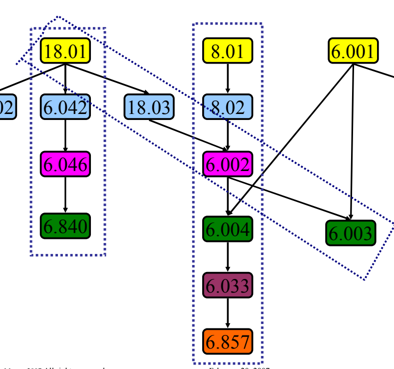
Chains

Set of successive prereqs

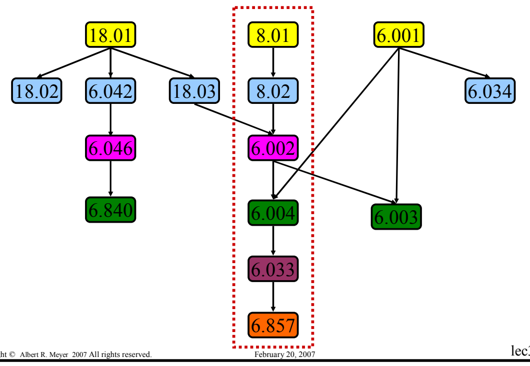
-- must be taken in order.

(subjects said to be *comparable*)

Some Chains



Maximum Length Chain



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How many terms to graduate?

- 6 terms are **necessary** to complete the curriculum
- *and* **sufficient** (if you can take unlimited subjects per term...)

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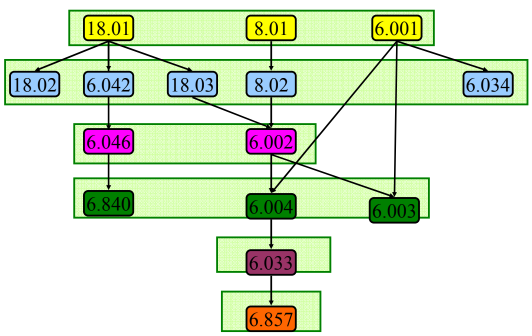
Sufficient…



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Parallel Processing Time

min parallel time = max chain size

required # processors

(term load in this case)

$\le \underbrace{\text{max antichain size}}_{\text{5 in this case}}$

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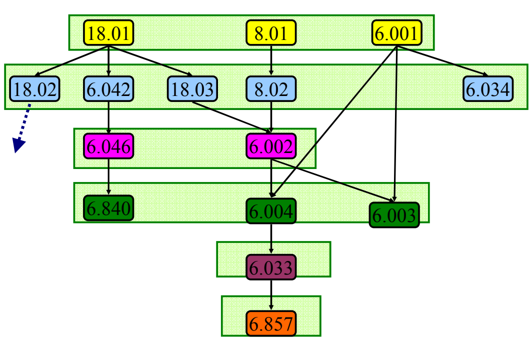
But can reduce the term load



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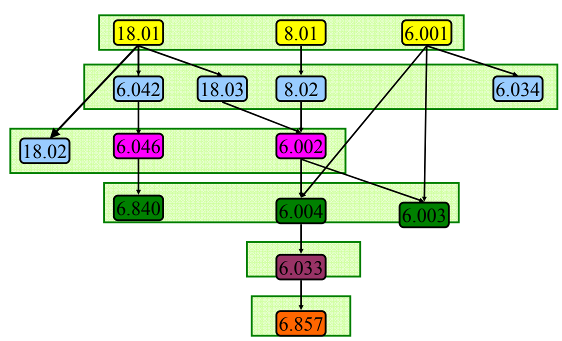
Only 4 Subjects per Term



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3 Subjects per Term Possible

A 3-course term is necessary

- **15** subjects
- max chain size = **6**
- size of *some* block must be

$$\geq \lceil 15/6 \rceil = 3.$$

∴ to finish in 6 terms, must take ≥3 subjects some term

Parallel Task Scheduling

Theorem: If the longest chain has size t, then the subjects can be *partitioned* into

t successive antichains,

with all prerequisites of an antichain in earlier ones.

Dilworth's Lemma

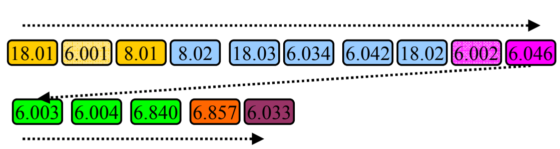
Prereq's among n subjects has

- a chain of size $\geq t$, ***or***
- or an antichain of size $\geq \left\lceil \frac{n}{t} \right\rceil$

for all $1 \leq t \leq n$.

A Leisurely Schedule

Graduate taking only 1 subject/term? Sure,



a *topological sort*

Team Problem

Problem 1

Partial Orders

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Subject Prerequisites

If subjects c, d are *mutual prereq's*:

$c \longrightarrow d$, and $d \longrightarrow c$

then no one can graduate!

Comm. on Curricula ensures:

if $c \longrightarrow d$, then $\neg(d \longrightarrow c)$

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Asymmetry

Binary relation, R, on set A, is *asymmetric* iff

aRb implies $\neg(bRa)$

for all $a,b \in A$

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Transitivity

Binary relation, R, on set A, is transitive:

aRb and bRc implies aRc

for all $a,b,c \in A$.

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Strict Partial Orders

Binary relation, R, on set A, is *a strict partial order* iff

- it is transitive and
- asymmetric

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Some Partial Orders

- $\leq$ on the Integers
- $<$ on the Reals

total

- $\subseteq$ on Sets (subset)
- $\subset$ on Sets (*proper* subset)

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Total **Order on** *A*

Partial Order, *R*, such that

$$aRb \quad or \quad bRa$$

for all $a \neq b \in A$

Partial Orders

- $y << x$ (*much* less than)

(say, $y + 2 \leq x$)

$\neg\, [3 << 4]$ and $\neg\, [4 << 3]$

incomparable

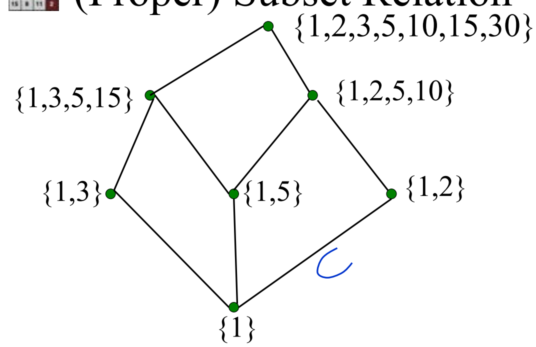
Representing Partial Orders

The subset relation,

$$\subseteq$$

on sets is the *canonical example* of weak partial order

(Proper) Subset Relation

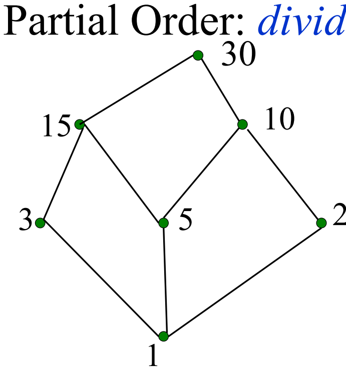


Partial Order: *divides*

a divides b iff

$ka = b$ for some $k \in \mathbb{N}$

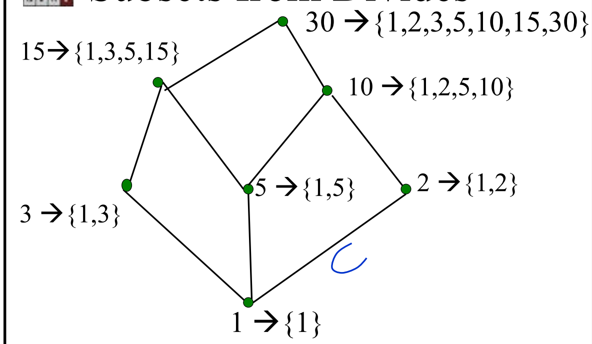
Partial Order: *divides*



Divides & Subset

same "shape"

Subsets from Divides



Team Problems

Problems 2−4