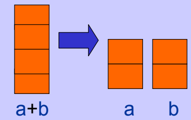
Mathematics for Computer Science
MIT 6.042J/18.062J

Strong Induction
Well Ordering Principle

 February 23, 2007

Unstacking game



- Start: a stack of boxes
- Move: split any stack into two stacks of sizes *a*,*b*>0
- Scoring: *ab* points
- Keep moving: until stuck
- Overall score: sum of move scores

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Analyzing the Stacking Game

Claim: Every way of unstacking gives the same score.

From stack of size *n*, what score?

Must be

$$(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$$

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Analyzing the Stacking Game

Claim: Starting with size *n* stack, final score will be

$$\frac{n(n-1)}{2}$$

Proof: by Induction with *Claim*(*n*) as hypothesis

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Proving the Claim by Induction

Base case $n = 0$:

$$\text{score} = 0 = \frac{0(0-1)}{2}$$

Claim(0) is O.K.

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Proving the Claim by Induction

Inductive step. assume for *n*-stack, and then prove *C*(*n*+1):

$$(n+1)\text{-stack score} = \frac{(n+1)n}{2}$$

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Proving the Claim by Induction

Inductive step.

Case $n+1 = 1$. verify for 1-stack:

$$\text{score} = 0 = \frac{1(1-1)}{2}$$

$C(1)$ is O.K.

Proving the Claim by Induction

Inductive step.

Case $n+1 > 1$. So split into an a-stack and b-stack, where $a + b = n + 1$.

$(a + b)$-stack score = ab + a-stack score + b-stack score

Proving the Claim by Induction

by induction:

a-stack score = $\frac{a(a-1)}{2}$

b-stack score = $\frac{b(b-1)}{2}$

Proving the Claim by Induction

total $(a + b)$-stack score =

$$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} =$$

$$\frac{(a+b)((a+b)-1)}{2} = \frac{(n+1)n}{2}$$

so $C(n+1)$ is O.K.

We're done!

Proving the Claim by Induction

Wait: we *assumed* $C(a)$ and $C(b)$ where $1 \le a, b \le n$.

But by induction can only assume $C(n)$

Proving the Claim by Induction

the fix:

revise the induction hypothesis to

$$Q(n) ::= \forall m \le n.\ C(m)$$

Proving the Claim by Induction

Proof goes through fine using $Q(n)$ instead of $C(n)$. So it's OK to assume

$$C(m) \text{ for all } m \le n$$

to prove $C(n+1)$.

Strong Induction

Prove $P(0)$. Then prove $P(n+1)$ assuming *all* of

$$P(0), P(1), \ldots, P(n)$$

(instead of just $P(n)$).

Conclude $\forall n.P(n)$

Strong vs. Ordinary

Why use Strong?
-- Convenience:
no need to include
"$\forall m \le n$" all over.

Postage by Strong Induction

available stamps: 

5¢ 3¢

Theorem:
Can form any amount ≥ 8¢

Prove by strong induction on n.
$P(n)$ ::= can form $(n+8)$¢.

(Picture source:http://site17585.dellhost.com/lsj/facts/s_events.htm http://www.frbsf.org/currency/civilwar/stamps/s150.html)

Postage by Strong Induction

Base case ($n = 0$):

$(0+8)$¢: 

Postage by Strong Induction

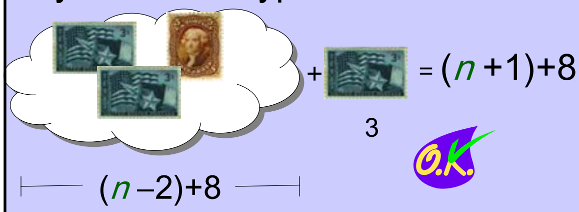
Inductive Step:
assume $(m+8)$¢ for $0 \le m \le n$,
then prove $((n+1)+8)$¢

cases:
$n+1 = 1$, 9¢: 

$n+1 = 2$, 10¢: 

Postage by Strong Induction

case $n+1 \geq 3$: let $m = n - 2$.
now $n \geq m \geq 0$, so
by induction hypothesis have:



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Team Problem

Problem 1

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Well Ordering Principle

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Well Ordering principle

Every nonempty set of
nonnegative integers
has a
least element.

Familiar? Now you mention it, Yes.
Obvious? Yes.
Trivial? Yes. But watch out:

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Well Ordering principle

Every nonempty set of
nonnegative rationals
has a
least element.

NO!

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Well Ordering principle

Every nonempty set of
~~nonnegative~~ integers
has a
least element.

NO!

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$\sqrt{2}$ proof used Well Ordering

Thm: $\sqrt{2}$ is irrational

Proof: suppose $\sqrt{2} = \frac{m}{n}$

…can always find such *m*, *n* without common factors…

why always?

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Proof using Well Ordering

By *WOP*, ∃ minimum |*m*| s.t.

$$\sqrt{2} = \frac{m}{n}. \quad \text{so} \quad \sqrt{2} = \frac{m_0}{n_0}$$

where $|m_0|$ is minimum.

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Proof using Well Ordering

but if m_0, n_0 had common factor $c > 1$, then

$$\sqrt{2} = \frac{m_0 / c}{n_0 / c}$$

and $|m_0 / c| < |m_0|$

contradicting minimality of $|m_0|$

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Well Ordering Principle Proofs

To prove ``$\forall n \in \mathbb{N}.\ P(n)$'' using WOP:

- Define the set of *counterexamples* $C ::= \{n \in \mathbb{N} \mid \neg P(n)\}$
- Assume *C* is not empty.
- By WOP, have minimum element $m_0 \in C$.
- Reach a contradiction (*somehow*) – usually by finding a member of *C* that is $< m_0$.
- Conclude no counterexamples exist. QED

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Team Problem

Problem 2

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