Mathematics for Computer Science
MIT 6.042J/18.062J

Predicate Logic
Quantifiers $\forall$, $\exists$

Predicates

Predicates are
Propositions with variables

Example:

$$P(x,y) \underbrace{::=}_{\text{“is defined to be”}} x + 2 = y$$

Predicates

$$P(x, y) ::= [x + 2 = y]$$

$x = 1$ and $y = 3$: $P(1,3)$ is true

$x = 1$ and $y = 4$: $P(1,4)$ is false

$\neg P(1,4)$ is true

Quantifiers

$\forall x$ For ALL x

$\exists y$ There EXISTS some y

Quantifiers

x, y range over ***Domain of Discourse***

$$\forall x \exists y.\ x < y$$

<u>*Domain*</u>	<u>*Truth value*</u>
integers $\mathbb{Z}$	True
positive integers $\mathbb{Z}^+$	True
negative integers $\mathbb{Z}^-$	False
negative reals $\mathbb{R}^-$	True

$\forall\exists$ versus $\exists\forall$

$\forall a \in$ **attack** $\exists d \in$ **defense.**
d **protects against** a

For every attack, I have a defense:

against **MYDOOM**, use Defender
against **ILOVEYOU**, use Norton
against **BABLAS**, use Zonealarm …

$\forall\exists$ is expensive!

$\exists\forall$

$\exists d \in$ defense $\forall a \in$ attack. d protects against a

I have *one* defense good against every attack.

Example: d is MITviruscan, protects against *all* viruses

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So $\exists\forall$ is better here

$\boxed{\exists d \in \text{defense } \forall a \in \text{attack. } d \text{ protects against } a}$

I have *one* defense good against every attack.

That's what we want!

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Math vs. English

Poet: "All that $\underbrace{\text{glitters}}^{G}$ is not $\underbrace{\text{gold}}^{Au}$."

$$\cancel{\forall x. G(x) \rightarrow \neg Au(x)}$$

No!: gold glitters like gold

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Math vs. English

Poet:

"All that glitters is not necessarily gold."

$$\boxed{\neg[\forall x. G(x) \rightarrow Au(x)]}$$

(Poetic license)

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Math vs. English

Poet: "There is a season for every purpose under heaven"

$\exists \boldsymbol{s} \in$ Season $\forall \boldsymbol{p} \in$ Purpose. $\boldsymbol{s}$ is for $\boldsymbol{p}$

So some season, say Spring, is good for all Purposes?

NO, Spring is no good for snow shoveling

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Poetic license again:

Poet: "There is a season for every purpose under heaven"

$\underbrace{\exists \boldsymbol{s} \in \text{Season}}\ \underbrace{\forall \boldsymbol{p} \in \text{Purpose}}$. $\boldsymbol{s}$ is for $\boldsymbol{p}$

Poet's meaning flips the quantiers

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Poetic license again:

Poet: "There is a season for every purpose under heaven"

$\forall p \in \text{Purpose}\ \exists s \in \text{Season}.\ s$ is for p

for snow shoveling, *Winter* is good
for planting, *Spring* is good
for leaf watching, *Fall* is good
etc.

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Team Problems

Problems 1 & 2

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Propositional Validity

$$(A \to B) \vee (B \to A)$$

True ***no matter what*** the truth values of A and B are

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Predicate Calculus Validity

$$\forall z\,[Q(z) \wedge P(z)] \to [\forall x.Q(x) \wedge \forall y.P(y)]$$

True ***no matter what***
- the Domain is,
- or the predicates are.

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Not Valid

$$\forall z\,[Q(z) \vee P(z)] \to [\forall x.Q(x) \vee \forall y.P(y)]$$

Proof: Give countermodel, where $\forall z\,[Q(z) \vee P(z)]$ is true, but $\forall x.Q(x) \vee \forall y.P(y)$ is false.

Namely, let domain $::= \{e, \pi\}$,
$Q(z) ::= [z = e]$,
$P(z) ::= [z = \pi]$.

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Predicate Calculus Validity

$$\forall z\,[Q(z) \wedge P(z)] \to [\forall x.Q(x) \wedge \forall y.P(y)]$$

Proof strategy: We assume

$\forall z\,[Q(z) \wedge P(z)]$

to prove

$\forall x.Q(x) \wedge \forall y.P(y)$

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Universal Generalization (UG)

$$\frac{A \to R(c)}{A \to \forall x.R(x)}$$

providing c does not occur in A

Validities

$$\forall z\,[Q(z) \wedge P(z)] \to [\forall x.Q(x) \wedge \forall y.P(y)]$$

Proof: Assume $\forall z\,[Q(z) \wedge P(z)]$.
So $Q(z) \wedge P(z)$ holds for all z in the domain.
Now let c be some domain element. So $Q(c) \wedge P(c)$ holds, and therefore $Q(c)$ by itself holds.
But c could have been any element of the domain.
So we conclude $\forall x.Q(x)$. (by ***UG***)
We conclude $\forall y.P(y)$ similarly. Therefore,
$\forall x.Q(x) \wedge \forall y.P(y)$ QED.

More Validities

$$\forall x[P(x) \vee A] \leftrightarrow [\forall x.P(x)] \vee A$$

providing x does not occur in A

$$[\neg\forall x.P(x)] \leftrightarrow [\exists x.\neg P(x)]$$

(version of DeMorgan)

Team Problems

Problems

4 & 3