Mathematics for Computer Science

MIT 6.042J/18.062J

Propositional Logic, II

Proof by Cases

Proof by Contradiction

Proof by Truth Tables

DeMorgan's Law

$\neg(P \vee Q)$ is *equivalent* to $\overline{P} \wedge \overline{Q}$

P	Q	$\neg(P \vee Q)$	
T	T	F	T
T	F	F	T
F	T	F	T
F	F	T	F

$\overline{P}$	$\overline{Q}$	$\overline{P} \wedge \overline{Q}$
F	F	F
F	T	F
T	F	F
T	T	T

Proof by Deductions

A student is trying to prove that propositions *P*, *Q*, and *R* are all true. She proceeds as follows.

First, she proves three facts:

- *P* implies *Q*
- Q implies *R*
- *R* implies *P*.

Then she concludes,

``Thus *P*, *Q*, and *R* are obviously all true."

Proposed Deduction Rule

From: *P* implies *Q*, *Q* implies *R*, *R* implies *P*

Conclude: *P*, *Q*, and *R* are true.

$$\frac{(P \to Q),\ (Q \to R),\ (R \to P)}{P \wedge Q \wedge R}$$

Sound Rule?

Conclusion true whenever all antecedents true.

$\underbrace{P \to Q \quad Q \to R \quad R \to P}_{\textit{Antecedents}} \qquad \underbrace{P \wedge Q \wedge R}_{\textit{Conclusion}}$

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

$\underbrace{P \to Q \quad Q \to R \quad R \to P}_{\textit{Antecedents}} \qquad \underbrace{P \wedge Q \wedge R}_{\textit{Conclusion}}$

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$	$P \wedge Q \wedge R$	sound?
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	
T	F	T	F	T	T	F	
T	F	F	F	T	T	F	
F	T	T	T	T	F	F	
F	T	F	T	F	T	F	
F	F	T	T	T	F	F	
F	F	F	T	T	T	F	

Antecedents ($P \to Q$, $Q \to R$, $R \to P$) *Conclusion* ($P \wedge Q \wedge R$)

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$	$P \wedge Q \wedge R$	sound?
T	T	T	T	T	T	T	
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$	$P \wedge Q \wedge R$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$	$P \wedge Q \wedge R$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	

Sound Rule?

Conclusion true whenever all antecedents true.

P	Q	R	$P \to Q$	$Q \to R$	$R \to P$	$P \wedge Q \wedge R$	sound?
T	T	T	T	T	T	T	OK
T	T	F	T	F	T	F	OK
T	F	T	F	T	T	F	OK
T	F	F	F	T	T	F	OK
F	T	T	T	T	F	F	OK
F	T	F	T	F	T	F	OK
F	F	T	T	T	F	F	OK
F	F	F	T	T	T	F	NOT OK!

Reasoning by Cases

Quicker proof of unsoundness than from truth tables

Quicker by Cases

$$\frac{P \to Q,\ Q \to R,\ R \to P}{P \wedge Q \wedge R}$$

Case 1: *P* is true. Now, if antecedents are true, then *Q* must be true (because *P* implies *Q*). Then *R* must be true (because *Q* implies *R*). So the conclusion $P \wedge Q \wedge R$ is true.

This case is OK.

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Quicker by Cases

$$\frac{P \to Q,\ Q \to R,\ R \to P}{P \wedge Q \wedge R}$$

Case 2: *P* is false. To make antecedents true, *R* must be false (because *R* implies *P*), so *Q* must be false (because *Q* implies *R*). This assignment does make the antecedents true, but the conclusion $P \wedge Q \wedge R$ is (very) false.

This case is not OK.

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Goldbach Conjecture

Every even integer greater than 2 is the sum of two primes.

Evidence:

$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 5 + 3$$
$$\vdots$$
$$20 = ? \quad 13 + 7$$

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Goldbach Conjecture

True for all even numbers with up to 13 digits! (Rosen, p.182)

It remains an OPEN problem: no counterexample, no proof.

UNTIL NOW!…

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Goldbach Conjecture

The answer is on my desk!

(Proof by Cases)

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Team Problem

Problem 1

Feb. 12, 2007 lec 2M.18

Proof by Contradiction

$$\frac{\overline{P} \rightarrow \mathbf{F}}{P}$$

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

- Suppose $\sqrt{2}$ was rational.
- Choose *m*, *n* integers without common prime factors (always possible) such that

$$\sqrt{2} = \frac{m}{n}$$

- Show that *m* & *n* are both even, a **contradiction**!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

$\sqrt{2} = \frac{m}{n}$

$\sqrt{2}n = m$

$2n^2 = m^2$

so *m* is even.

so can assume $m = 2l$

$m^2 = 4l^2$

$2n^2 = 4l^2$

$n^2 = 2l^2$

so *n* is even.

Quickie

Proof assumes that

If m^2 is even, then *m* is even.

Why!

Team Problem

Problems 2 & 3