Mathematics for Computer Science

MIT 6.042J/18.062J

Sets & Functions

What is a Set?

Informally:

A ***set*** is a collection of mathematical objects, with the collection treated as a single mathematical object.

(*This is circular of course:*

what's a collection?)

Some sets

real numbers, $\mathbb{R}$

complex numbers, $\mathbb{C}$

integers, $\mathbb{Z}$

empty set, $\varnothing$

set of all subsets of integers, pow($\mathbb{Z}$)

the power set

Some sets

$\{7, \text{“Albert R.”}, \pi/2, \mathrm{T}\}$

A set with 4 elements: two numbers, a string, and a Boolean value.

Same as

$\{\text{“Albert R.”}, 7, \mathrm{T}, \pi/2\}$

-- order doesn't matter

Membership

$x \in A$ x is an element of A

$\pi/2 \in \{7, \text{“Albert R.”}, \pi/2, \mathrm{T}\}$

$\pi/3 \notin \{7, \text{“Albert R.”}, \pi/2, \mathrm{T}\}$

$14/2 \in \{7, \text{“Albert R.”}, \pi/2, \mathrm{T}\}$

Synonyms for Membership

$x \in A$ x is a member of A

x is in A

Examples:

$7 \in \mathbb{Z}$ $2/3 \notin \mathbb{Z}$ $\mathbb{Z} \in \text{pow}(\mathbb{R})$

In or Not In

An element is in or not in a set:

$\{7, \pi/2, 7\}$ is same as $\{7, \pi/2\}$

(No notion of being in the set more than once)

Containment

$A \subseteq B$ A is a subset of B

A is contained in B

Every element of A is also an element of B.

$\mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{R} \subseteq \mathbb{C}$, $\{3\} \subseteq \{5,7,3\}$

$\varnothing \subseteq$ every set, $A \subseteq A$

Defining Sets

The set of elements, x, in A

such that $P(x)$ is true.

$$\{x \in A \mid P(x)\}$$

Defining Sets

The set of even integers:

$$\{n \in \mathbb{Z} \mid n \text{ is even}\}$$

New sets from old

union:

$A \cup B ::= \{x \mid (x \in A) \vee (x \in B)\}$

intersection:

$A \cap B ::= \{x \mid x \in A \wedge x \in B\}$

difference:

$A - B ::= \{x \mid (x \in A) \wedge (x \notin B)\}$

power set

$$\text{pow}(A) ::= \{S \mid S \subseteq A\}$$

$$\text{pow}(\{a,b\}) = \{\{a,b\},\{a\},\{b\},\varnothing\}$$

Quickie

What is Pow(∅)?

Ans: {∅}

What is Pow(Pow(∅))?

Ans: {{∅}, ∅}

Russell's Paradox

Let $W ::= \{S \in \text{Sets} \mid S \notin S\}$

so $S \in W \leftrightarrow S \notin S$

Let S be W and reach a contradiction:

$W \in W \leftrightarrow W \notin W$

Russell's Paradox

The fallacy: **W is not a set!**

No set is a member of itself, so W = the collection of all sets, which is **not** a set!

Team Problems

Problems 1 & 3

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Functions

$f : A \to B$

function, f, from set A to set B associates an element, $f(a) \in B$ with an element $a \in A$.

Example: f is the string-length function: $f(\text{“aabd”}) = 4$

f: Strings $\to \mathbb{N}$

The *domain* of f is the set of strings.

The *codomain* of f is the set of nonegative integers

$$g(x,y) ::= \frac{1}{x-y}$$

$$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

domain(g) = all pairs of reals

codomain(g)= all reals

But g is partial:

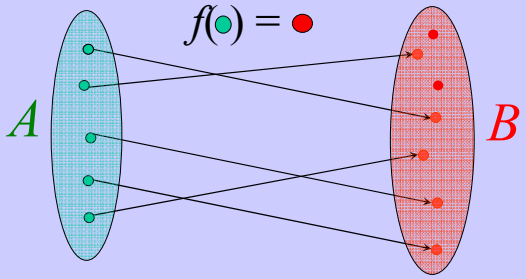
not defined on pairs (r,r)

Total functions

$f : A \to B$ is total iff every element of A is assigned a B-value by f

Total functions

exactly 1 arrow out

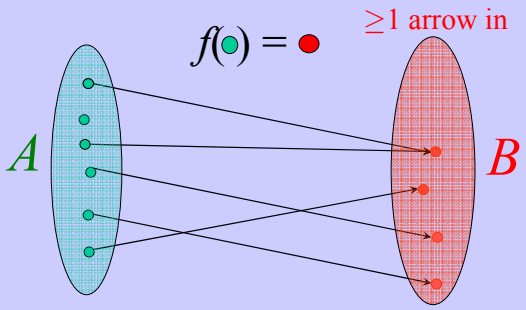


Surjections

$f : A \to B$ is a *surjection* iff every element of B is f of something

Surjection

≥1 arrow in



Mapping Rule

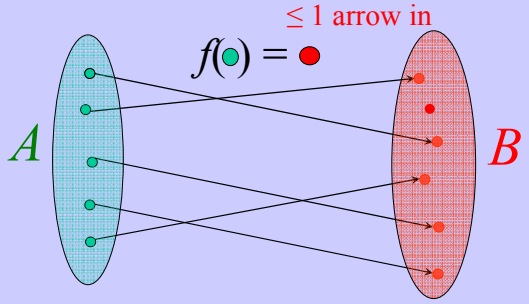
surjection $A \to B$ implies

$$|A| \geq |B|$$

Injections

$f : A \to B$ is an *injection* iff every element of B is f of *at most* 1 thing

Injections



Mapping Rule

injection $A \to B$ implies

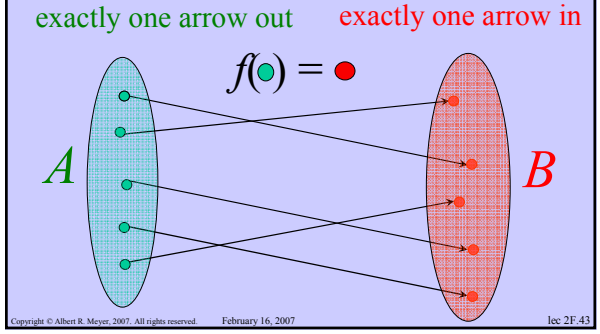
$$|A| \leq |B|$$

total

Bijections

$f : A \to B$ is a *bijection* iff it is all those good things: total, onto, and 1-1

Bijections



Mapping Rule

bijection $A \to B$ implies

$$|A| = |B|$$

Team Problems

Problem 2