

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Sets & Functions

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What is a Set?

Informally:

A *set* is a collection of mathematical objects, with the collection treated as a single mathematical object.

(This is *circular* of course:
what's a collection?)

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Some sets

real numbers, \mathbb{R}
complex numbers, \mathbb{C}
integers, \mathbb{Z}
empty set, \emptyset
set of all subsets of integers, $\text{pow}(\mathbb{Z})$
the **power** set

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Some sets

$\{7, \text{"Albert R."}, \pi/2, \mathbf{T}\}$

A set with 4 **elements**: two numbers, a string, and a Boolean value.

Same as

$\{\text{"Albert R."}, 7, \mathbf{T}, \pi/2\}$

-- order doesn't matter

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Membership

$x \in A$ x is an **element** of A

$\pi/2 \in \{7, \text{"Albert R."}, \pi/2, \mathbf{T}\}$

$\pi/3 \notin \{7, \text{"Albert R."}, \pi/2, \mathbf{T}\}$

$14/2 \in \{7, \text{"Albert R."}, \pi/2, \mathbf{T}\}$

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Synonyms for Membership

$x \in A$ x is a **member** of A

x is **in** A

Examples:

$7 \in \mathbb{Z}$ $2/3 \notin \mathbb{Z}$ $\mathbb{Z} \in \text{pow}(\mathbb{R})$

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In or Not In

An element is **in** or **not in** a set:

$\{7, \pi/2, 7\}$ is same as $\{7, \pi/2\}$

(No notion of being in the set more than once)

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Containment

$A \subseteq B$ A is a **subset** of B
 A is **contained in** B

Every element of A is also an element of B .

$\mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{R} \subseteq \mathbb{C}$, $\{3\} \subseteq \{5, 7, 3\}$

$\emptyset \subseteq$ every set, $A \subseteq A$

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Defining Sets

The **set of elements**, x , in A
such that $P(x)$ is true.

$\{x \in A \mid P(x)\}$

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Defining Sets

The set of **even** integers:
 $\{n \in \mathbb{Z} \mid n \text{ is even}\}$

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New sets from old

union:

$A \cup B ::= \{x \mid (x \in A) \vee (x \in B)\}$

intersection:

$A \cap B ::= \{x \mid x \in A \wedge x \in B\}$

difference:

$A - B ::= \{x \mid (x \in A) \wedge (x \notin B)\}$

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power set

$\text{pow}(A) ::= \{S \mid S \subseteq A\}$

$\text{pow}(\{a, b\}) = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$

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Quickie

What is $\text{Pow}(\emptyset)$?

Ans: $\{\emptyset\}$

What is $\text{Pow}(\text{Pow}(\emptyset))$?

Ans: $\{\{\emptyset\}, \emptyset\}$

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Russell's Paradox

Let $W := \{S \in \text{Sets} \mid S \notin S\}$

so $S \in W \leftrightarrow S \notin S$

Let S be W and reach a contradiction:

$W \in W \leftrightarrow W \notin W$

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Russell's Paradox

The fallacy: **W is not a set!**

No set is a member of itself, so
 W = the collection of all sets,
 which is **not** a set!

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Team Problems

Problems

1 & 3

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Mathematics for Computer Science

MIT 6.042J/18.062J

Functions

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$f : A \rightarrow B$

function, f , from set A to set B
 associates an element, $f(a) \in B$
 with an element $a \in A$.

Example: f is the string-length
 function: $f(\text{"aabd"}) = 4$

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$$f: \text{Strings} \rightarrow \mathbb{N}$$

The *domain* of f is the set of strings.

The *codomain* of f is the set of nonnegative integers

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$$g(x, y) ::= \frac{1}{x - y}$$

$$g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

domain(g) = all pairs of reals

codomain(g) = all reals

But g is *partial*:

not defined on pairs (r, r)

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Total functions

$f: A \rightarrow B$ is *total* iff every element of A is assigned a B -value by f

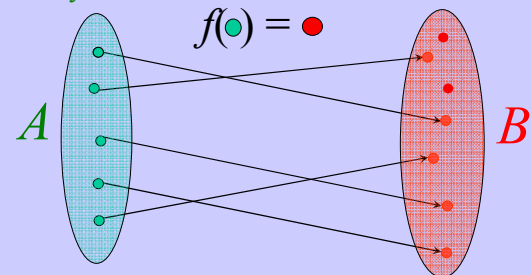
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Total functions

exactly 1 arrow out



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Surjections

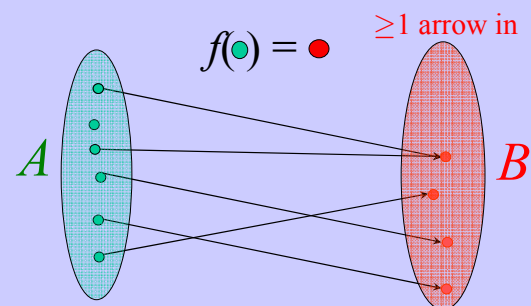
$f: A \rightarrow B$ is a *surjection* iff every element of B is f of something

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Surjection



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Mapping Rule

surjection $A \rightarrow B$ implies
 $|A| \geq |B|$

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Injections

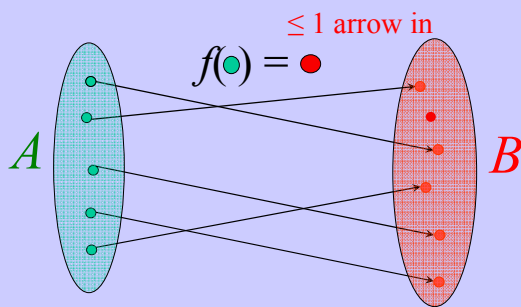
$f : A \rightarrow B$ is an *injection*
 iff every element of B is
 f of *at most* 1 thing

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Injections



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Mapping Rule

injection $A \rightarrow B$ implies
 $|A| \leq |B|$
total

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Bijections

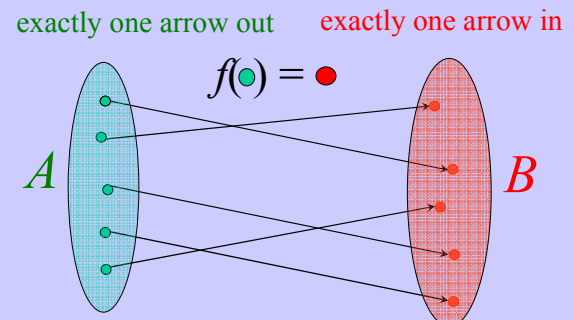
$f : A \rightarrow B$ is a *bijection* iff
 it is all those good things:
 total, onto, and 1-1

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Bijections



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Mapping Rule

bijection $A \rightarrow B$ implies
 $|A| = |B|$

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Team Problems

Problem 2

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