Mathematics for Computer Science

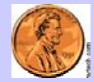
MIT 6.042J/18.062J

Deviation from the Mean

 May 14, 2006 lec 14M.1

Don't expect the Expectation!

Toss 101 fair coins.



E[#Heads] = 50.5

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Don't expect the Expectation!

Pr{*exactly* 50.5 Heads} = 0

Pr{*exactly* 50 Heads} < 1/13

Pr{50.5 $\pm$1 Heads} < 1/7

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Don't expect the Expectation!

Toss 1001 fair coins.

E[#Heads] = 500.5

Pr{#H = 500} < 1/39

Pr{#H = 500.5$\pm$1 } < 1/19

smaller

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Within a % of the mean?

Toss 1001 fair coins.

of 1001

Pr{#H = 500 $\pm$ 1%}

= Pr{#H = 500 $\pm$ 10}

$\approx$ 0.49

not so bad

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Giving *Meaning* to the "Mean"

Let μ ::= E[R]

- What is $\underbrace{\Pr\{R \textit{ far from } \mu\}}_{\Pr\{|R-\mu|>x\}}$?
- R's *average deviation*?

$$E[\,|R-\mu|\,]?$$

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Two Dice with Same Mean

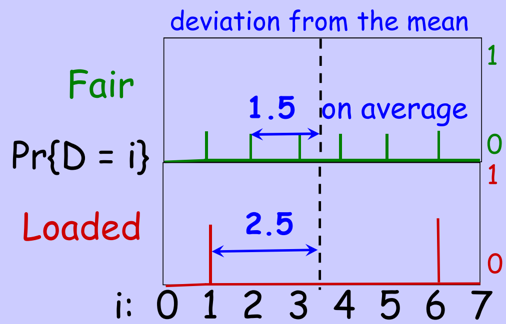
Fair Die

- $E[D_1] = 3.5$

Loaded Die throwing only 1 & 6:

- $E[D_2] = (1+6)/2 = 3.5$ also!

Two Dice with Same Mean



Dice have Different Deviations

Fair Die:

$E[\ |D_1 - \mu|\ ] = 1.5$

Loaded Die:

$E[\ |D_2 - \mu|\ ] = 2.5$

Giving *Meaning* to the "Mean"

The mean alone is not a good predictor of R's behavior. We generally need more about its distribution, especially probable deviation from its mean.

Example: IQ

IQ measure was constructed so that

average IQ = 100.

What fraction of the people can *possibly* have an IQ ≥ 300?

IQ Higher than 300?

Fraction f with IQ ≥ 300 adds ≥ $300f$ to average, so 100 = avg IQ ≥ $300f$:

$$f \leq 100/300 = 1/3$$

IQ Higher than 300?

At most 1/3 of people have IQ ≥ 300

$$\Pr\{IQ \geq 300\} \leq \frac{E[IQ]}{300}$$

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IQ Higher than x?

In general,

$$\Pr\{IQ \geq x\} \leq \frac{100}{x}$$

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IQ Higher than x?

Besides mean = 100,
we used *only one fact about the distribution* of IQ:

IQ is always nonnegative

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Markov Bound

If R is nonnegative, then

$$\boxed{\Pr\{R \geq x\} \leq \frac{E[R]}{x}}$$

for $x > 0$.

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Markov Bound

- Weak
- Obvious
- Useful anyway

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IQ ≥ 300, again

Suppose we are *given* that IQ is always ≥ 40?
Get a better bound on fraction f with IQ ≥ 300, by considering

IQ − 40

since this is now ≥ 0.

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IQ ≥ 300, again

f contributes (300-40)**f** to the average of IQ−40, so

$$60 = E[IQ-40] \geq 260f$$

$$f \leq 60/260 = 3/13$$

Better bound from Markov by *shifting* R to have 0 as minimum

Improving the Markov Bound

$$\Pr\{|R-\mu| \geq x\}$$
$$= \Pr\{(R-\mu)^2 \geq x^2\}$$

by Markov:

$$\leq \frac{E[(R-\mu)^2]}{x^2}$$

variance of R

Chebyshev Bound

$$\boxed{\Pr\{|R-\mu| \geq x\} \leq \frac{\mathrm{Var}[R]}{x^2}}$$

Variance of an Indicator

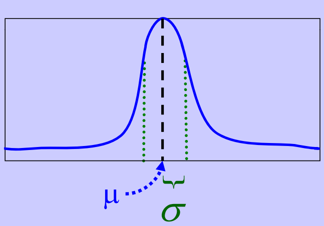
I an indicator with E[I]=p:

$$\boxed{\mathrm{Var}[I]} ::= E\left[(I-p)^2\right]$$
$$= E\left[I^2 - 2pI + p^2\right]$$
$$= E\left[I^2\right] - 2p\,E[I] + p^2$$
$$= p - 2p^2 + p^2 = \boxed{p(1-p)}.$$

Variance and Standard Deviation

$$\sigma_R ::= \sqrt{\mathrm{Var}[R]}$$

PDF_R



Standard Deviation

$$\boxed{\Pr\{|R-\mu| \geq x\} \leq \frac{\sigma^2}{x^2}}$$

R *probably not many* σ's from μ:

further than	σ	Pr ≤ 1
	2σ	Pr ≤ 1/4
	3σ	Pr ≤ 1/9
	4σ	Pr ≤ 1/16

Calculating Variance

$$\text{Var}[aR + b] = a^2\,\text{Var}[R]$$

$$\text{Var}[R] = \text{E}\left[R^2\right] - \text{E}^2[R]$$

(simple proofs applying linearity of expectation to the def of variance)

Calculating Variance

$$\text{Var}[R_1 + R_2 + \cdots + R_n] =$$
$$\text{Var}[R_1] + \text{Var}[R_2] + \cdots + \text{Var}[R_n]$$

providing $R_1, R_2, \ldots, R_n$ are *pairwise independent*

Calculating Variance

Pairwise Independent Additivity

similar proof using linearity of expectation & def of variance

Team Problems

Problems

1–5