Mathematics for Computer Science
MIT 6.042J/18.062J

Conditional Probability & Independence

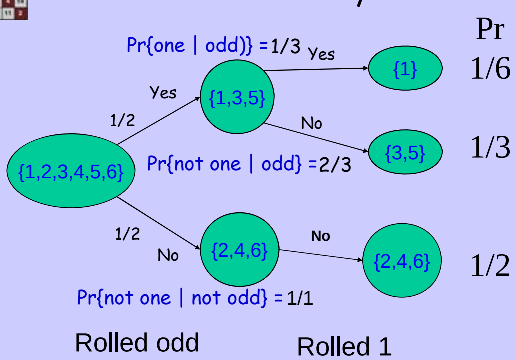
Conditional Probability: Dice

Pr{die rolled 1} = 1/|{1,2,3,4,5,6}|
= 1/6.

"Knowledge" changes probabilities:

Pr{die rolled 1 **knowing** that die rolled odd number}
= 1/|{1,3,5}|
= 1/3.

Conditional Probability: Dice



Conditional Probability

$\Pr\{A \mid B\}$ is the prob. of event A, given that event B has occurred

$$\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

Product Rule

$$\Pr\{A \cap B\} = \Pr\{A \mid B\}\Pr\{B\}$$

Conditional Probability: Monty Hall

Pr{prize at 1 | Goat at 2} = **1/2**

Really!

Conditional Probability: Monty Hall

Pr{prize at 1 | Goat at 2} = **1/2**

Outcomes: Really!

(Prize Door, Picked Door, Carol door)

[Goat at 2] =

{(1,1,2),(1,1,3),(1,2,3),(1,3,2)
(3,3,1),(3,3,2),(3,1,2),(3,2,1)}

Conditional Probability: Monty Hall

Pr{prize at 1 | Goat at 2} = **1/2**

Really! Outcomes:

(Prize Door, Picked Door, Carol door)

[Goat at 2] =

{(1,1,2),(1,1,3),(1,2,3),(1,3,2)
(3,3,1),(3,3,2),(3,1,2),(3,2,1)}

Conditional Probability: Monty Hall

Pr{prize at 1 | Carol opens 2} = **1/2**

Outcomes:

(Prize Door, Picked Door, Carol door)

[Carol opens 2] =

{(1,1,2),(1,3,2),
(3,3,2),(3,1,2)}

Conditional Probability: Monty Hall

Pr{prize at 1 | Carol opens 2} = **1/2**

Outcomes:

(Prize Door, Picked Door, Carol door)

[Carol opens 2] =

{(1,1,2),(1,3,2),
(3,3,2),(3,1,2)}

Conditional Probability: Monty Hall

This suggests the contestant may as well stick, since the probability is 1/2 *given what he knows* when he chooses. But wait: contestant knows more than door opened by Carol -- also knows: which door he chose himself!

Conditional Probability: Monty Hall

Pr{prize at 1 | picked 1 &
Carol opens 2} = **1/3**

[picked 1 & Carol opens 2] =

{(1,1,2),(3,1,2)}

Conditional Probability: Monty Hall

Pr{prize at 1 | picked 1 & Carol opens 2} = **1/3**

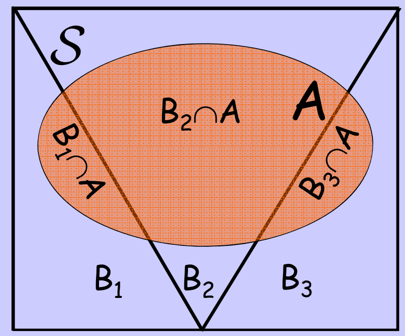
[picked 1 & Carol opens 2] =

$\{\underbrace{(1,1,2)}_{Pr=1/18},\underbrace{(3,1,2)}_{Pr=1/9}\}$

$$\frac{1/18}{1/18+1/9} = \frac{1}{3}$$

 May 2, 2007

Law of Total Probability



 May 2, 2007

Law of Total Probability

$A = (B_1 \cap A) \cup (B_2 \cap A) \cup (B_3 \cap A)$

$\Pr\{A\} = \Pr\{B_1 \cap A\} + \Pr\{B_2 \cap A\} + \Pr\{B_3 \cap A\}$

$= \Pr\{A \mid B_1\} \cdot \Pr\{B_1\} + \Pr\{A \mid B_2\} \cdot \Pr\{B_2\} + \Pr\{A \mid B_3\} \cdot \Pr\{B_3\}$

 May 2, 2007

Team Problems

Problems 1 & 2

 May 2, 2007

Definitions of Independence

Definition 1:

Events A and B are independent iff

$\Pr\{A\} = \Pr\{A \mid B\}$.

Definition 2:

Events A and B are independent iff

$\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}$.

 May 2, 2007

Definitions of Independence

Equivalent:

$\Pr\{A\} = \Pr\{A \mid B\}$ iff

$\Pr\{A\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$ iff

$\Pr\{A\} \cdot \Pr\{B\} = \Pr\{A \cap B\}$.

 May 2, 2007

Definitions of Independence

Note: need Pr{B} ≠ 0 for Def. 1.
Def. 2 works even if 0:

$$\Pr\{A\}\cdot\Pr\{B\} = \Pr\{A\cap B\}$$

 May 2, 2007 lec 12W.21

The Birthday "Paradox"

Puzzle: n students in a room. Probability that two have the same birthday (month, day) for n = 2, 10, 23, 30, 107?

 May 2, 2007 lec 12W.22

The Birthday "Paradox"

- So with 10 students have 10/365 ≈ 1/30 chance 2 have same b'day?
  Not really, it's more like 1/10.
- With 30 students, maybe 3·(30/365) ≈ 1/3 chance?
  No, it's more than 2 to 1!

 May 2, 2007 lec 12W.23

The Birthday "Paradox"

Let's stop guessing and figure it out. Let's assume 6.042 students are equally likely to have each of 365 possible birthdays.

 May 2, 2007 lec 12W.24

The Birthday "Paradox"

Choose 2 students at random.
Pr{2 students have *same* b'day}

$$= \frac{1}{365}$$

 May 2, 2007 lec 12W.25

The Birthday "Paradox"

Pr{2 students b'days *differ*}

$$= 1 - \frac{1}{365}$$

 May 2, 2007 lec 12W.26

The Birthday "Paradox"

Choose another 2 students
independently of first two.

Pr{neither pair has same birthday}
= Pr{1st pair's b'days differ and
2nd pair's b'days differ}
= Pr{1st pair's b'days differ} ×
Pr{2nd pair's b'days differ}

 May 2, 2007 lec 12W.31

The Birthday "Paradox"

Pr{both pairs' b'days differ}

$$= \left(1 - \frac{1}{365}\right)^2$$

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The Birthday "Paradox"

Choose another 253 pairs of students
independently of first pairs.

Pr{*no* pair has same birthday}

$$= \left(1 - \frac{1}{365}\right)^{253} \approx \frac{1}{2}$$

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The Birthday "Paradox"

But with n = 23 *students*,
have $\binom{23}{2} = 253$ *pairs*
of students.

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The Birthday "Paradox"

So, with 23 students
Pr{*no* pair has same b'day}

$$\approx \frac{1}{2}$$

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The Birthday "Paradox"

With 140 students
Pr{*no* pair has same b'day}

$$\approx \left(1 - \frac{1}{365}\right)^{\binom{140}{2}} = \left(1 - \frac{1}{365}\right)^{9730}$$

 May 2, 2007 lec 12W.36

The Birthday "Paradox"

With 140 students
Pr{*no* pair has same b'day}

$$= \left(1 - \frac{1}{365}\right)^{365\left(\frac{9730}{365}\right)} \leq e^{-\left(\frac{9730}{365}\right)}$$

$$\leq \frac{1}{300{,}000{,}000{,}000}$$

The Birthday "Paradox"

In fact, in a term with 6.042 enrollment of 140, we found 17 pairs with same birthday (and 2 triples)

The Birthday "Paradox"

Wait! Whether one pair of students has the same birthday is not independent of other pairs: if (Joy, Tim) have same b'day, and (Tim, Mike) do too, then

Pr{(Joy,Mike) same b'day} = 1.

The Birthday "Paradox"

But this dependence actually makes same b'day pairs *more* likely, so our value for

Pr{no matches}

is a valid *upper* bound.

The Birthday "Paradox"

…and when #students << # b'days (for example, 23 << 365), our bound is tight, because pairs w/same b'day not likely to overlap.

Team Problems

Problems

3 & 4