Mathematics for Computer Science

MIT 6.042J/18.062J

Introduction to Random Variables

Guess the Bigger Number

Team 1:

- Write different integers between 0 and 7 on two pieces of paper
- Show to Team 2 face down

Team 2:

- Expose one paper and look at number
- Either *stick* or *switch* to other number

Team 2 wins if ends with **larger** number

Guess the Bigger Number

Try it out!

Strategy for Team 2

Choose papers with equal probability. If exposed number is "small" then switch; otherwise stick.

"small" means $\leq$ threshold Z.

Z is random integer, $0 \leq Z < 7$.

Analysis of Team 2 Strategy

Case (low $\leq Z <$ high):

Team 2 wins in this case, so

Pr{Team 2 wins} = 1

and Pr{this case} $\geq \frac{1}{7}$

Analysis of Team 2 Strategy

Case (high $\leq Z$):

Team 2 will switch, so wins iff low card gets exposed.

Pr{Team 2 wins} = $\frac{1}{2}$

Analysis of Team 2 Strategy

Case (Z < low):

Team 2 will stick, so

wins iff high card gets exposed.

$$\Pr\{\text{Team 2 wins}\} = \frac{1}{2}$$

Analysis of Team 2 Strategy

So 1/7 of time, sure win.

Rest of time, 50/50 win, so

$\Pr\{\text{Team 2 wins}\} \geq$

$$\frac{1}{7}\cdot 1+\frac{6}{7}\cdot\frac{1}{2}=\frac{4}{7}>\frac{1}{2}$$

Analysis of Team 2 Strategy

Does not matter
what Team 1 does!!

Team Problem

Problem 1

How can Team 1 guarantee

$$\Pr\{\text{Team 2 wins}\} \leq \frac{4}{7}$$

whatever Team 2 does?

Random Variables

Informally: an RV is a number produced by a *random process:*

- number of larger card
- number of smaller card
- number of exposed card
- threshold variable Z

What is a Random Variable?

Formally,

$$R : \mathcal{S} \to \mathbb{R}$$

Sample space (usually)

Intro to Random Variables

Example: Flip three fair coins.

C ::= number of heads (**C**ount).

$$M ::= \begin{cases} 1 & \text{if all Match,} \\ 0 & \text{otherwise.} \end{cases}$$

Intro to Random Variables

Specify events using values of variables.

- $[C = 1]$ is the event "exactly 1 head"

  $\Pr\{C = 1\} = 3/8$
- $\Pr\{C \geq 1\} = 7/8$
- $\Pr\{C \cdot M > 0\} = \Pr\{M>0 \text{ and } C>0\}$

  $= \Pr\{\text{all heads}\} = 1/8$

Independent Variables

Random variables R,S

are independent iff

$[R = a]$, $[S = b]$

are independent *events*

for all numbers a, b.

Independent Variables

Alternative version 1: R,S independent iff

$$\Pr\{R = a \mid S = b\} = \Pr\{R = a\}.$$

Alternative version 2:

$$\Pr\{R = a \text{ and } S = b\} = \Pr\{R = a\} \cdot \Pr\{S = b\}.$$

Independent Variables

Tell me:

Are C and M independent?

Independent Variables

H_1 ::= indicator for Head on coin 1

H_2 ::= indicator for Head on coin 2

$P ::= H_1 \oplus H_1$ (mod 2 sum).

any 2 of them are independent:

$\Pr\{P=0 \mid H_2=a\} = 1/2 = \Pr\{P=0\}$, etc.

But any 2 determine the 3rd one,

so the 3 *together* are not really independent.

Independent Variables

Pairwise Independence:

$\Pr\{A_i=a_i \text{ and } A_j=a_j\} =$

$\Pr\{A_i=a_i\} \cdot \Pr\{A_j=a_j\}$ all $i \neq j$.

Mutual Independence:

$\Pr\{A_1=a_1 \text{ and } A_2=a_2 \text{ and } \cdots A_n=a_n\} =$

$\Pr\{A_1=a_1\} \cdot \Pr\{A_2=a_2\} \cdots \Pr\{A_n=a_n\}$.

Independent Variables

k-wise Independence:

any k of the variables are mutually independent

(so 2-wise = pairwise)

Independent Variables

Pairwise Independence sufficient for major applications (in later lecture).

Good to know, since pairwise holds in important cases where mutual does not.

Team Problems

Problems 2&3