

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Binomial Theorem, Combinatorial Proof

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Polynomials Express Choices & Outcomes

$$(\text{red tie} + \text{orange tie} + \text{yellow tie}) (\text{orange tie} + \text{gray tie}) =$$

$$\text{red tie orange tie} + \text{red tie gray tie} + \text{orange tie orange tie} + \text{orange tie gray tie} + \text{yellow tie orange tie} + \text{yellow tie gray tie}$$

Products of Sum = Sums of Products

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Expression for c_k ?

$$(1+X)^n =$$

$$c_0 + c_1X + c_2X^2 + \dots + c_nX^n$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Expression for c_k ?

$$(1+X)^n \quad \text{n times}$$

$$= (1+X)(1+X)(1+X)(1+X)\dots(1+X)$$

multiplying gives 2^n product terms:
 $11\dots 1 + X11X\dots 1 + 1XX\dots 1X1 + \dots + XX\dots X$
 a term corresponds to selecting 1 or X from each of the n factors.

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Expression for c_k ?

$$(1+X)^n \quad \text{n times}$$

$$= (1+X)(1+X)(1+X)(1+X)\dots(1+X)$$

the X^k coeff, c_k , is number of terms where exactly k X's were selected.

$$c_k = \binom{n}{k}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Binomial Formula

$$(1+X)^n =$$

$$\binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \dots + \binom{n}{k}X^k + \dots + \binom{n}{n}X^n$$

binomial coefficients

binomial expression

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Binomial Formula

$$\begin{aligned}
 (1+X)^0 &= 1 \\
 (1+X)^1 &= 1 + 1X \\
 (1+X)^2 &= 1 + 2X + 1X^2 \\
 (1+X)^3 &= 1 + 3X + 3X^2 + 1X^3 \\
 (1+X)^4 &= 1 + 4X + 6X^2 + 4X^3 + 1X^4
 \end{aligned}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Binomial Formula

$$\begin{aligned}
 (X + Y)^n &= \\
 &\binom{n}{0}Y^n + \binom{n}{1}XY^{n-1} + \binom{n}{2}X^2Y^{n-2} + \\
 &\dots + \binom{n}{k}X^kY^{n-k} + \dots + \binom{n}{n}X^n
 \end{aligned}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Binomial Formula

$$(X + Y)^n = \sum_{k=0}^n \binom{n}{k} X^k Y^{n-k}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomials

What is the coefficient of
EMSTY
in the expansion of
(E + M + S + T + Y)⁵ ?

5!

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lec 11M.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomials

What is the coefficient of
EMS³TY
in the expansion of
(E + M + S + T + Y)⁷ ?

The number of ways to
rearrange the letters in
the word
SYSTEMS

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lec 11M.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Applying the BOOKKEEPER rule

What is the coefficient of
EMS³TY
in the expansion of
(E + M + S + T + Y)⁷ ?

7!
1! 1! 3! 1! 1!

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

$$\binom{7}{1,1,3,1,1} ::= \frac{7!}{1! 1! 3! 1! 1!}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

$$\binom{n}{r_1, r_2, \dots, r_k} ::= \frac{n!}{r_1! r_2! \dots r_k!}$$

$$= 0 \quad \text{if } r_1 + r_2 + \dots + r_k \neq n$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

What is the coefficient of
 BA^3N^2
 in the expansion of
 $(B + A + N)^6$?

The number of ways to
 rearrange the letters in
 the word
 BANANA

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

What is the coefficient of
 BA^3N^2
 in the expansion of
 $(B + A + N)^6$?

$$\binom{6}{1,3,2}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

What is the coefficient of
 $X_1^{r_1} X_2^{r_2} X_3^{r_3} \dots X_k^{r_k}$
 in the expansion of
 $(X_1 + X_2 + X_3 + \dots + X_k)^n$?

$$\binom{n}{r_1, r_2, r_3, \dots, r_k}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Multinomial Coefficients

Binomial a special case:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

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6	9	13	7
12		10	5
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15	8	11	2

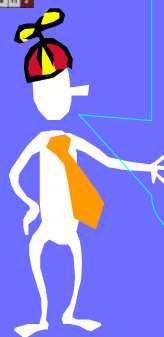
The Multinomial Formula

$$(X_1 + X_2 + \dots + X_k)^n = \sum_{\substack{r_1, r_2, \dots, r_k \\ \sum r_i = n}} \binom{n}{r_1, r_2, \dots, r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \dots X_k^{r_k}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2



More next week
about how
polynomials
encode counting
questions!

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof: routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1}$$

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lec 11M.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Consider subsets of $\{1, \dots, n\}$

size k subsets =
size k subsets that contain a 1
+ # size k subsets that do not contain a 1

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lec 11M.24

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Consider subsets of $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\substack{\text{\# size } k \\ \text{subsets}}} = \underbrace{\binom{n-1}{k}}_{\substack{\text{\# size } k \\ \text{subsets}}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\text{\# size } k-1 \\ \text{subsets}}}$$

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lec 11M.25

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Consider subsets of $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\substack{\text{\# size } k \\ \text{subsets}}} = \underbrace{\binom{n-1}{k}}_{\substack{\text{\# size } k \\ \text{subsets}}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\text{\# size } k-1 \\ \text{subsets:} \\ \text{with no } 1}}$$

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lec 11M.26

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Consider subsets of $\{1, \dots, n\}$

$$\underbrace{\binom{n}{k}}_{\substack{\text{\# size } k \\ \text{subsets}}} = \underbrace{\binom{n-1}{k}}_{\substack{\text{\# size } k \\ \text{subsets:} \\ \text{with no } 1}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\text{\# size } k \\ \text{subsets:} \\ \text{with a } 1}}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

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lec 11M.28

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Consider subsets of $\{1, \dots, n, 1, \dots, n\}$

$$\sum_{i=0}^n \binom{n}{i}^2 = \underbrace{\binom{2n}{n}}_{\substack{\text{\# size } n \\ \text{subsets}}}$$

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lec 11M.29

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

$$\begin{aligned} \text{LHS} &= \sum_{i=0}^n \binom{n}{i}^2 \\ &= \sum_{i=0}^n \binom{n}{i} \binom{n}{n-i} \end{aligned}$$

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lec 11M.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\text{\# size } i \\ \text{red subsets}}} \binom{n}{n-i}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

LHS =

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\text{\# size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\text{\# size } n-i \\ \text{black subsets}}}$$

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lec 11M.32

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

$$\sum_{i=0}^n \underbrace{\binom{n}{i}}_{\substack{\text{\# size } i \\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\text{\# size } n-i \\ \text{black subsets}}}$$

So LHS = # size n subsets
of $\{1, \dots, n, 1, \dots, n\}$
by the Sum Rule

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lec 11M.33

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Combinatorial Proof

Therefore
LHS = # size n subsets = RHS

$$\sum_{i=0}^n \binom{n}{i}^2 = \binom{2n}{n}$$

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6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Team Problems

Problems
1–4

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