Mathematics for Computer Science

MIT 6.042J/18.062J

Binomial Theorem, Combinatorial Proof

Polynomials Express Choices & Outcomes

Products of Sum = Sums of Products

Expression for c_k?

$$(1+X)^n =$$

$$c_0 + c_1X + c_2X^2 + \ldots + c_nX^n$$

Expression for c_k?

$$(1+X)^n = \underbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}_{n \text{ times}}$$

multiplying gives 2^n product terms:

$11\cdots1 + X11X\cdots X1 + 1XX\cdots 1X1 + \cdots + XX\cdots X$

a term corresponds to selecting 1 or X from each of the n factors.

Expression for c_k?

$$(1+X)^n = \underbrace{(1+X)(1+X)(1+X)(1+X)\ldots(1+X)}_{n \text{ times}}$$

the X^k coeff, c_k, is number of terms where exactly k X's were selected.

$$c_k = \binom{n}{k}$$

The Binomial Formula

binomial expression

$$(1+X)^n =$$

$$\binom{n}{0} + \binom{n}{1}X + \binom{n}{2}X^2 + \ldots + \binom{n}{k}X^k + \ldots + \binom{n}{n}X^n$$

binomial coefficients

The Binomial Formula

$(1+X)^0 = 1$

$(1+X)^1 = 1 + 1X$

$(1+X)^2 = 1 + 2X + 1X^2$

$(1+X)^3 = 1 + 3X + 3X^2 + 1X^3$

$(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + 1X^4$

The Binomial Formula

$$(X+Y)^n =$$

$$\binom{n}{0}Y^n + \binom{n}{1}XY^{n-1} + \binom{n}{2}X^2Y^{n-2} +$$

$$\ldots + \binom{n}{k}X^kY^{n-k} + \ldots + \binom{n}{n}X^n$$

The Binomial Formula

$$\boxed{(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}}$$

Multinomials

What is the coefficient of
EMSTY
in the expansion of
$(E + M + S + T + Y)^5$?

5!

Multinomials

What is the coefficient of
EMS^3TY
in the expansion of
$(E + M + S + T + Y)^7$?

The number of ways to rearrange the letters in the word
SYSTEMS

Applying the BOOKKEEPER rule

What is the coefficient of
EMS^3TY
in the expansion of
$(E + M + S + T + Y)^7$?

$$\frac{7!}{1!\ 1!\ 3!\ 1!\ 1!}$$

Multinomial Coefficients

$$\binom{7}{1,1,3,1,1} ::= \frac{7!}{1!\ 1!\ 3!\ 1!\ 1!}$$

Multinomial Coefficients

$$\boxed{\binom{n}{r_1, r_2, \ldots, r_k} ::= \frac{n!}{r_1! r_2! \ldots r_k!}}$$

$$= 0 \qquad \text{if } r_1 + r_2 + \ldots + r_k \neq n$$

Multinomial Coefficients

What is the coefficient of

BA^3N^2

in the expansion of

$(B + A + N)^6$?

The number of ways to rearrange the letters in the word BANANA

Multinomial Coefficients

What is the coefficient of

BA^3N^2

in the expansion of

$(B + A + N)^6$?

$$\binom{6}{1,3,2}$$

Multinomial Coefficients

What is the coefficient of

$X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$

in the expansion of

$(X_1+X_2+X_3+\ldots+X_k)^n$?

$$\boxed{\binom{n}{r_1, r_2, r_3, \ldots, r_k}}$$

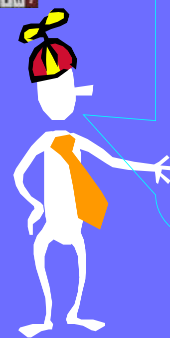
Multinomial Coefficients

Binomial a special case:

$$\binom{n}{k} = \binom{n}{k, n-k}$$

The Multinomial Formula

$$(X_1+X_2+\ldots+X_k)^n = \sum_{\substack{r_1,r_2,\ldots,r_k \\ \sum r_i = n}} \binom{n}{r_1,r_2,\ldots,r_k} X_1^{r_1} X_2^{r_2} X_3^{r_3} \ldots X_k^{r_k}$$



More next week about how polynomials encode counting questions!

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof: routine, using

$$\binom{n}{k} ::= \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}$$

Combinatorial Proof

Consider subsets of {1,…,n}

\# size k subsets =
  \# size k subsets that contain a 1
  \+ # size k subsets that do not contain a 1

Combinatorial Proof

Consider subsets of {1, …, n}

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof

Consider subsets of {1, …, n}

$$\underbrace{\binom{n}{k}}_{\text{\# size k subsets}} = \underbrace{\binom{n-1}{k}}_{\text{\# size k subsets: with no 1}} + \underbrace{\binom{n-1}{k-1}}$$

Combinatorial Proof

Consider subsets of {1, …, n}

$$\underbrace{\binom{n}{k}}_{\substack{\#\text{ size } k\\ \text{subsets}}} = \underbrace{\binom{n-1}{k}}_{\substack{\#\text{ size } k\\ \text{subsets:}\\ \text{with no } 1}} + \underbrace{\binom{n-1}{k-1}}_{\substack{\#\text{ size } k\\ \text{subsets:}\\ \text{with a } 1}}$$

Combinatorial Proof

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Combinatorial Proof

Consider subsets of {1,…,n,1,….,n}

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \underbrace{\binom{2n}{n}}_{\substack{\#\text{ size } n\\ \text{subsets}}}$$

Combinatorial Proof

$$\text{LHS} = \sum_{i=0}^{n} \binom{n}{i}^2$$

$$= \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$$

Combinatorial Proof

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size } i\\ \text{red subsets}}} \binom{n}{n-i}$$

Combinatorial Proof

LHS =

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size } i\\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\#\text{ size } n-i\\ \text{black subsets}}}$$

Combinatorial Proof

$$\sum_{i=0}^{n} \underbrace{\binom{n}{i}}_{\substack{\#\text{ size } i\\ \text{red subsets}}} \underbrace{\binom{n}{n-i}}_{\substack{\#\text{ size } n\text{-}i\\ \text{black subsets}}}$$

So LHS = # size n subsets of {1,…,n,1,….,n} by the Sum Rule

Combinatorial Proof

Therefore
LHS = # size n subsets = RHS

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

Team Problems

Problems 1—4