Mathematics for Computer Science

MIT 6.042J/18.062J

Generating Functions for Recurrences

The Rabbit Population

- A mature boy/girl rabbit pair reproduces every month.
- Rabbits mature after one month.

w_n::= # newborn pairs after n months

r_n::= # reproducing pairs after n months

- Start with a newborn pair: $w_0 = 1$

  $r_0 = 0$

The Rabbit Population

w_n::= # newborn pairs after n months

r_n::= # reproducing pairs after n months

$$r_1 = 1$$

$$r_n = r_{n-1} + w_{n-1}$$

$$w_n = r_{n-1} \text{ so}$$

$$r_n = r_{n-1} + r_{n-2}$$

The Rabbit Population

$$r_n = r_{n-1} + r_{n-2}$$

It was Fibonacci who was studying rabbit population growth

Generating Function for Rabbits

$$\begin{aligned} R(x) &::= r_0 + r_1x + r_2x^2 + r_3x^3 + \cdots \\ -xR(x) &= -r_0x - r_1x^2 - r_2x^3 - \cdots \\ -x^2R(x) &= -r_0x^2 - r_1x^3 - \cdots \\ \hline & \ \ 0 \end{aligned}$$

Generating Function for Rabbits

$$\begin{aligned} R(x) &::= r_0 + r_1x + r_2x^2 + r_3x^3 + \cdots \\ -xR(x) &= -r_0x - r_1x^2 - r_2x^3 - \cdots \\ -x^2R(x) &= -r_0x^2 - r_1x^3 - \cdots \\ \hline & \ \ 0 \qquad 0 \qquad \cdots \end{aligned}$$

Generating Function for Rabbits

R(x)::= $r_0 + r_1 x$
$-xR(x) = -r_0 x$
$-x^2R(x) =$

$R(x) - xR(x) - x^2R(x) =$
$r_0 + r_1 x - r_0 x = x$

Generating Function for Rabbits

$$R(x) = \frac{x}{1 - x - x^2}$$

Now find closed form for r_n:

Closed Form for r_n

$$R(x) = \frac{x}{1 - x - x^2}$$
$$= \frac{x}{(1 - \alpha x)(1 - \beta x)}$$

α, β are 1/roots of $1-x-x^2$

Closed Form for r_n

$$R(x) = \frac{x}{(1 - \alpha x)(1 - \beta x)}$$
$$= \frac{a}{1 - \alpha x} + \frac{b}{1 - \beta x}$$

so

$$r_n = a\alpha^n + b\beta^n$$

Closed Form for r_n

from quadratic formula:

$$\alpha = \frac{1 + \sqrt{5}}{2}$$
$$\beta = \frac{1 - \sqrt{5}}{2}$$

Closed Form for r_n

$$x = a(1 - \beta x) + b(1 - \alpha x)$$

$x = 1/\beta$: $1/\beta = b(1 - \alpha/\beta)$
$b = 1/(\beta - \alpha)$
likewise $a = 1/(\alpha - \beta)$

Closed Form for r_n

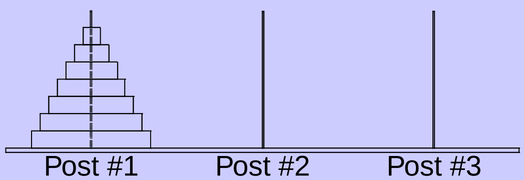
$$r_n = a\alpha^n + b\beta^n$$
$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Closed Form for r_n

$$r_n = \left\lfloor \frac{((1+\sqrt{5})/2)^n}{\sqrt{5}} \right\rceil$$

$$(1.61)^n = o(r_n)$$
$$r_n = o((1.62)^n)$$

Towers of Hanoi



$\text{Move}_{1,2}(n)$::= $\text{Move}_{1,3}(n-1)$;
big disk 1→2;
$\text{Move}_{3,2}(n-1)$

Towers of Hanoi

s_n::=# steps by $\text{Move}_{1,2}(n)$

$$s_n = 2s_{n-1} + 1$$
$$s_0 = 0$$

Hanoi Generating Function

$S(x)$::= $s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots$
$-2xS(x) = -2s_0x - 2s_1x^2 - 2s_2x^3 - \cdots$
$-x/(1-x) = -1\cdot x - 1\cdot x^2 - 1\cdot x^3 - \cdots$

0 0 0 …

Hanoi Generating Function

$S(x) = s_0 = 0$
$-2xS(x)$
$-x/(1-x)$

$$S(x) = \frac{x}{(1-x)(1-2x)}$$

Hanoi Generating Function

$$S(x) = \frac{x}{(1-x)(1-2x)}$$

$$= \frac{a}{1-x} + \frac{b}{1-2x}$$

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Hanoi Generating Function

$$x = a(1-2x) + b(1-x)$$

for x = 1: 1 = a(-1), so

a = -1

x = 1/2: 1/2 = b(1/2), so

b = 1

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Hanoi Generating Function

$$S(x) = \frac{1}{1-2x} - \frac{1}{1-x}$$

so $s_n = 2^n - 1$

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Team Problems

Problems

1 & 2

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