

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Generating Functions for Recurrences

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lec 11F.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Rabbit Population



- A mature boy/girl rabbit pair reproduces every month.
- Rabbits mature after one month.

$w_n ::=$ # newborn pairs after n months

$r_n ::=$ # reproducing pairs after n months

- Start with a newborn pair: $w_0 = 1$
 $r_0 = 0$

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lec 11F.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Rabbit Population



$w_n ::=$ # newborn pairs after n months

$r_n ::=$ # reproducing pairs after n months

$$r_1 = 1$$

$$r_n = r_{n-1} + w_{n-1}$$

$$w_n = r_{n-1} \text{ so}$$

$$r_n = r_{n-1} + r_{n-2}$$

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lec 11F.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

The Rabbit Population



$$r_n = r_{n-1} + r_{n-2}$$

It was Fibonacci who was studying rabbit population growth

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lec 11F.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Generating Function for Rabbits

$$R(x) ::= r_0 + r_1x + r_2x^2 + r_3x^3 + \dots$$

$$-xR(x) = -r_0x - r_1x^2 - r_2x^3 - \dots$$

$$-x^2R(x) = -r_0x^2 - r_1x^3 - \dots$$

0

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lec 11F.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Generating Function for Rabbits

$$R(x) ::= r_0 + r_1x + r_2x^2 + r_3x^3 + \dots$$

$$-xR(x) = -r_0x - r_1x^2 - r_2x^3 - \dots$$

$$-x^2R(x) = -r_0x^2 - r_1x^3 - \dots$$

0

0

...

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lec 11F.6

4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Generating Function for Rabbits

$$R(x) ::= r_0 + r_1 x$$

$$-xR(x) = -r_0 x$$

$$-x^2 R(x) =$$

$$R(x) - xR(x) - x^2 R(x) = r_0 + r_1 x - r_0 x = x$$

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4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Generating Function for Rabbits

$$R(x) = \frac{x}{1 - x - x^2}$$

Now find closed form for r_n :

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lec 11F.8

4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Closed Form for r_n

$$R(x) = \frac{x}{1 - x - x^2}$$

$$= \frac{x}{(1 - \alpha x)(1 - \beta x)}$$

α, β are 1/roots of $1 - x - x^2$

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lec 11F.9

4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Closed Form for r_n

$$R(x) = \frac{x}{(1 - \alpha x)(1 - \beta x)}$$

$$= \frac{a}{1 - \alpha x} + \frac{b}{1 - \beta x}$$

so

$$r_n = a\alpha^n + b\beta^n$$

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4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Closed Form for r_n from quadratic formula:

$$\alpha = \frac{1 + \sqrt{5}}{2}$$

$$\beta = \frac{1 - \sqrt{5}}{2}$$

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lec 11F.11

4	9	13	7
12		10	6
3	1	14	
15	8	11	5

Closed Form for r_n

$$x = a(1 - \beta x) + b(1 - \alpha x)$$

$$x=1/\beta: \quad 1/\beta = b(1 - \alpha/\beta)$$

$$b = 1/(\beta - \alpha)$$

$$\text{likewise} \quad a = 1/(\alpha - \beta)$$

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lec 11F.12

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Closed Form for r_n

$$r_n = a\alpha^n + b\beta^n$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

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lec 11F.13

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Closed Form for r_n

$$r_n = \left\lfloor \frac{((1 + \sqrt{5})/2)^n}{\sqrt{5}} \right\rfloor$$

$$(1.61)^n = o(r_n)$$

$$r_n = o((1.62)^n)$$

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lec 11F.14

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Towers of Hanoi



$\text{Move}_{1,2}(n) ::= \text{Move}_{1,3}(n-1);$
big disk $1 \rightarrow 2;$
 $\text{Move}_{3,2}(n-1)$

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lec 11F.15

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Towers of Hanoi

$s_n ::= \# \text{ steps by } \text{Move}_{1,2}(n)$

$$s_n = 2s_{n-1} + 1$$

$$s_0 = 0$$

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lec 11F.16

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Hanoi Generating Function

$$S(x) ::= s_0 + s_1x + s_2x^2 + s_3x^3 + \dots$$

$$-2xS(x) = -2s_0x - 2s_1x^2 - 2s_2x^3 - \dots$$

$$-x/(1-x) = -1 \cdot x^1 - 1 \cdot x^2 - 1 \cdot x^3 - \dots$$

0 0 0 ...

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lec 11F.17

4	9	13	7
12	10	6	
3	1	14	
15	8	11	5

Hanoi Generating Function

$$S(x) = s_0 = 0$$

$$-2xS(x)$$

$$-x/(1-x)$$

$$S(x) = \frac{x}{(1-x)(1-2x)}$$

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lec 11F.18

4	9	13	7
12	10	6	
3	1	8	14
15	5	11	2

Hanoi Generating Function

$$S(x) = \frac{x}{(1-x)(1-2x)}$$

$$= \frac{a}{1-x} + \frac{b}{1-2x}$$

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lec 11F.19

4	9	13	7
12	10	6	
3	1	8	14
15	5	11	2

Hanoi Generating Function

$$x = a(1-2x) + b(1-x)$$

for $x = 1$: $1 = a(-1)$, so

$$a = -1$$

$x = 1/2$: $1/2 = b(1/2)$, so

$$b = 1$$

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lec 11F.20

4	9	13	7
12	10	6	
3	1	8	14
15	5	11	2

Hanoi Generating Function

$$S(x) = \frac{1}{1-2x} - \frac{1}{1-x}$$

$$\text{so } s_n = 2^n - 1$$

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lec 11F.21

4	9	13	7
12	10	6	
3	1	8	14
15	5	11	2

Team Problems

Problems

1 & 2

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