Mathematics for Computer Science

MIT 6.042J/18.062J

Generalized Counting Rules

Pigeonhole Principle

Mapping Rule:

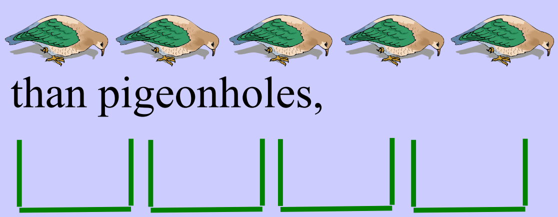
If ∃ injection A to B, then $|A| \leq |B|$.

If $|A| > |B|$, then no injection from A to B.

Pigeonhole Principle

If more pigeons

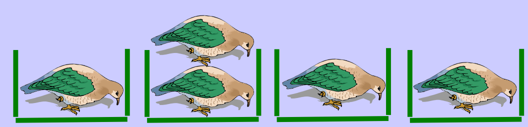
than pigeonholes,



Pigeonhole Principle

then some hole must have

$\geq$ two pigeons!



Example: 5 Card Draw

Set of 5 cards:

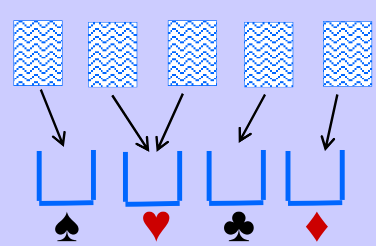
must have ≥ 2

with the same suit.

5 Card Draw

5 cards (pigeons)

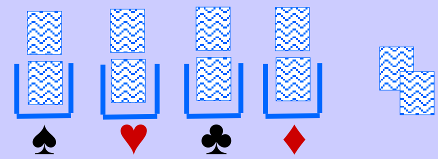
4 suits (holes)



10 Card Draw

10 cards: how many have the same suit?

10 Card Draw



Cannot have < 3 cards in every hole.

10 Card Draw

cards with same suit ≥ 3

$\left\lceil \frac{10}{4} \right\rceil = 3$ cards with same suit

"ceiling," means round up

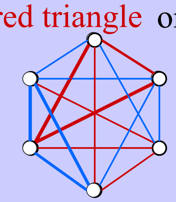
Generalized Pigeonhole Principle

If n pigeons and h holes, then some hole has at least

$\left\lceil \frac{n}{h} \right\rceil$ pigeons.

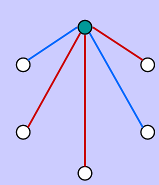
Colored Graph Claim

A 6-node complete graph with edges colored red or blue,
has *either* a red triangle or a blue triangle.



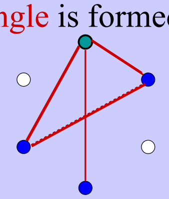
Colored Graph Claim: *proof*

Vertex of degree 5 has $\geq$
3 red or 3 blue incident edges.



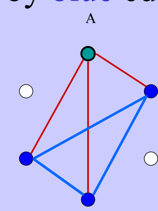
Proof

Say 3 red edges; if 2 of 3 endpoints are connected by red edge, then a red triangle is formed.



Proof

Otherwise, all 3 endponts are connected by blue edges



a blue triangle

Generalized Product Rule

How many sequences of 5 students in 6.042?

$S ::= 6.042$ students, $|S| = 101$

$\cancel{|\text{sequences of } 5| = 101^5 \; ?}$ NO!

We want

$|\text{sequences in } S^5 \text{ with no repeats.}|$

Generalized Product Rule

$|\text{sequences in } S^5 \text{ with no repeats.}|$

101 choices for 1st student,
100 choices for 2nd student,
99 choices for 3rd student,
98 choices for 4th student,
97 choices for 5th student

so $101 \cdot 100 \cdot 99 \cdot 98 \cdot 97 = \dfrac{101!}{96!}$

Generalized Product Rule

Q a set of length-k sequences. If there are:

n_1 possible 1st elements in sequences,

n_2 possible 2nd elements for each first entry,

n_3 possible 3rd elements for each 1st & 2nd,

$\vdots$

then, $|Q| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$

Division Rule

if function from A to B is k-to-1, then

$$|A| = k\,|B|$$

(generalizes the Bijection Rule)

Division Rule

#6.042 students =

$$\frac{\text{\#6.042 students' fingers}}{10}$$

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Counting Subsets

How many size 4 subsets of {1,2,…,13}?

Let A::= permutations of {1,2,…,13}

B::= size 4 subsets

map $a_1\, a_2\, a_3\, a_4\, a_5 \ldots a_{12}\, a_{13}$ to

$\{a_1, a_2, a_3, a_4\}$

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Counting Subsets

$a_2\, a_4\, a_3\, a_1\, a_5 \ldots a_{12}\, a_{13}$ also maps to

$\{a_1, a_2, a_3, a_4\}$

as does

$\underbrace{a_2\, a_4\, a_3\, a_1}_{4!}\, \underbrace{a_{13}\, a_{12} \ldots a_5}_{9!}$ $4! \cdot 9!$-to-1

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Counting Subsets

$$13! = |A| = 4!9!|B|$$

So number of 4 element subsets is

$$\binom{13}{4} ::= \frac{13!}{4!9!}$$

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Counting Subsets

Number of m element subsets of an n element set is

$$\binom{n}{m} ::= \frac{n!}{m!(n-m)!}$$

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Team Problems

Problems 1—3

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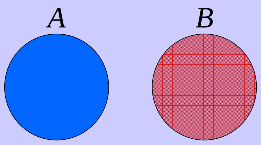
Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$

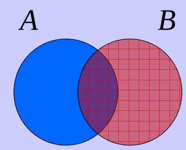


What if A and B are not disjoint?

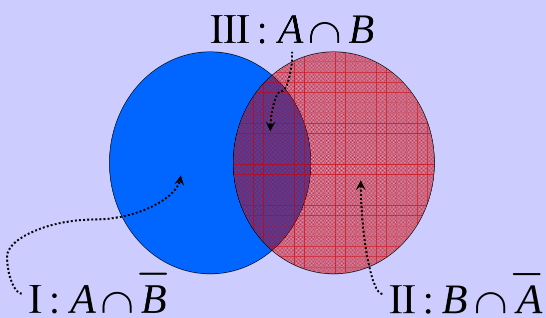
Inclusion-Exclusion (2 Sets)

For two arbitrary sets A and B

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

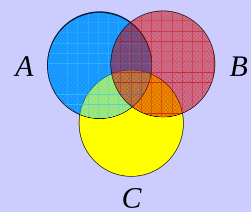


Inclusion-Exclusion (2 Sets)



Inclusion-Exclusion (3 Sets)

$$\begin{aligned}|A \cup B \cup C| = {} & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C|\end{aligned}$$



Team Problem

Problem 4