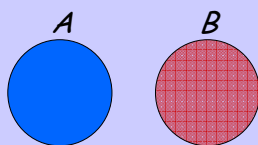


6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Sum Rule

$$|A \cup B| = |A| + |B|$$



for **disjoint** sets  $A, B$

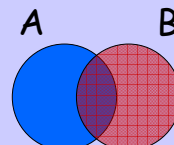
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lec 10F.1

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Sum Rule

$$|A \cup B| = ?$$



What if **not** disjoint?

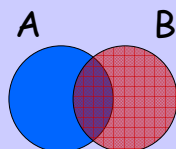
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lec 10F.2

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Inclusion-Exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$



What if **not** disjoint?

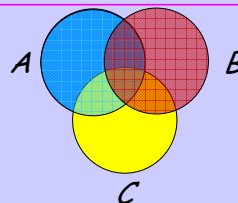
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lec 10F.3

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## Inclusion-Exclusion (3 Sets)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



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lec 10F.7

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

*Mathematics for Computer Science*

MIT 6.042J/18.062J

# Tricks with Counting & Matching

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lec 10F.10

6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

## The Magic Trick

- Students **choose** 5 cards
- Chiyoun **reveals** 4 of them
- Jessica **announces** 5<sup>th</sup> card

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lec 10F.11

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## The Magic Trick

# Let's do it!

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lec 10F.12

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## Chiyoun's Choices

- Decide **the order** of the 4 cards:  $4! = 24$  orderings  
-- but **48** cards remain
- Decide **which** 4 cards to list

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lec 10F.13

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## Match hands with 4-Card lists

5-card hands  
(no order)



?



4-card lists  
(ordered)



list must come  
from hand

×

Which one to pick?

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lec 10F.14

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## Match hands with 4-Card lists

5-card hands  
(no order)



?



4-card lists  
(ordered)



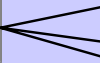
How can we ensure  
consistency?

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lec 10F.15

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## Match hands with 4-Card lists



$$\deg = \binom{5}{4} \times 4! = 120$$

$$\deg = 52 - 4 = 48$$



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lec 10F.21

4	9	13	7
12	10	6	1
3	1	14	15
16	8	11	5

## Match hands with 4-Card lists

The graph is  
**degree-constrained**  
so there is a match that  
Jessica and Chiyoun can use  
—even works for bigger decks

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lec 10F.22

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## A Memorable Matching?

$$\binom{52}{5} = 2,598,960 \text{ hands to match to lists}$$

How will Jessica & Chiyoun learn them?

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lec 10F.23

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## Magic Trick Revealed (I)

Among 5 cards chosen:

at least 2 have the same suit  
(Pigeonhole Principle)

Chiyoun lists one of them 1<sup>st</sup>

*Aha! The first card has the same suit as the hidden card!*

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lec 10F.24

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## Magic Trick Revealed (II)

How does Jessica figure out the value of the hidden card?

*Aha! Look at the order of the other 3 cards!*

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lec 10F.25

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## Magic Trick Revealed (II)

Fix ordering of the deck

$A\clubsuit < 2\clubsuit < 3\clubsuit < \dots < K\clubsuit <$

$A\diamondsuit < 2\diamondsuit < 3\diamondsuit < \dots < K\diamondsuit <$

$A\heartsuit < 2\heartsuit < 3\heartsuit < \dots < K\heartsuit <$

$A\spadesuit < 2\spadesuit < 3\spadesuit < \dots < K\spadesuit$

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lec 10F.26

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## Magic Trick Revealed (II)

Possible orders for the remaining 3 cards:

{ **SML, SLM, MSL, MLS, LSM, LMS** }

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lec 10F.27

4	9	13	7
12	10	6	1
3	5	8	14
15	11	16	2

## Magic Trick Revealed (II)

**Wait!** Only have 6 lists of the remaining 3 cards, but 12 possible hidden cards of the known suit!

*Of two cards with the same suit, choosing which to reveal can give 1 more bit of information!  
Aha!*

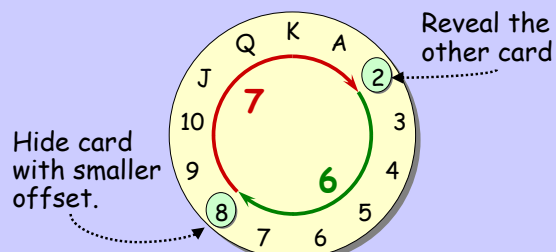
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lec 10F.28

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

### Clockwise Distance

The *smaller clockwise distance* between 2 card values is at most **6**:



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lec 10F.29

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

### Magic Trick Revealed (Finally)

- The first card determines the hidden suit ( $\spadesuit \heartsuit \diamondsuit \clubsuit$ ).
- Hidden value (A ... K) = first-card value + offset ( $\leq 6$ ).
- Offset given by order of remaining 3 cards:  
 $SML = 1, SLM = 2, MSL = 3,$   
 $MLS = 4, LSM = 5, LMS = 6.$

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lec 10F.30

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

### Example

First: Hidden:   
 Offset = 1 = **SML**:

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lec 10F.31

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

### Trick can't work with 4-card hands

Students can pick

$$\binom{52}{4} = 270,725$$

possible 4-card hands

Chiyoun can reveal

$$\frac{52!}{49!} = 132,600$$

possible 3-card lists

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lec 10F.32

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

Trick can't work with 4 cards hands so at least

$$\left\lceil \frac{270,225}{132,600} \right\rceil = 3$$

hands map to the **same list**  
 - Jessica can't tell which!

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lec 10F.33

4	9	13	7
12	10	6	5
3	1	8	14
15	11	16	2

### Team Problems

Problem 1  
 (& 2 & 3)

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lec 10F.35