Problem Set 9

Due: April 25

Reading:

- Week 9: Rules for Counting, §§3–6.
- Week 10: More Counting.

Problem 1. **(a)** Jellybeans of 6 different flavors are stored in 5 jars. There are 11 jellybeans of each flavor. Prove that some jar contains at least three jellybeans of one flavor and also at least three jellybeans of some other flavor.

(b) Prove that every finite undirected graph has two vertices of the same degree.

Problem 2. There is a robot that steps between integer positions in 3-dimensional space. Each step of the robot increments one coordinate and leaves the other two unchanged.

(a) Let m, n be nonnegative integers. Describe a bijection between the length-$(m + n)$ binary strings with exactly m ones and the paths the robot can take to go from $(0, 0, 0)$ to $(m, n, 0)$. How many such paths are there?

(b) How many different paths are there from point $(0, 0, 0)$ to $(12, 24, 36)$?

Problem 3. Suppose you have seven dice— each a different color of the rainbow; otherwise the dice are standard, with six faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is $(3, 1, 6, 1, 4, 5, 2)$ indicating that the red die showed a 3, the orange die showed 1, the yellow 6, the green 1, the blue 4, the indigo 5, and the violet 2.

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules. Then write a simple numerical expression for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R be the set of seven rainbow colors and S be the set $\{1, \ldots, 6\}$ of dice values.

Define $B ::= S_2 \times \{3, 4\} \times R_3$, where S_2 is the set of size 2 subsets of S, and R_3 is the set of size 3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is the set of two numbers that came up, whose second element is the number of times the smaller of the two numbers came up in the roll, and whose third element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 4, 2, 4) \in A$$

maps to the triple

$$(\{2, 4\}, 3, \{\text{yellow,green,indigo}\}) \in B.$$

Now by the Bijection rule $|A| = |B|$, and by the Product rule,

$$|B| = \binom{6}{2} \cdot 2 \cdot \binom{7}{3}.$$

(a) For how many rolls is the value on every die different?

(b) For how many rolls do two dice have the value 6 and the remaining five dice all have different values?

Example: $(6, 2, 6, 1, 3, 4, 5)$ is a roll of this type, but $(1, 1, 2, 6, 3, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(c) For how many rolls do two dice have the same value and the remaining five dice all have different values?

Example: $(4, 2, 4, 1, 3, 6, 5)$ is a roll of this type, but $(1, 1, 2, 6, 1, 4, 5)$ and $(6, 6, 1, 2, 4, 3, 4)$ are not.

(d) For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value?

Example: $(6, 1, 2, 1, 2, 6, 6)$ is a roll of this type, but $(4, 4, 4, 4, 1, 3, 5)$ and $(5, 5, 5, 6, 6, 1, 2)$ are not.

Problem 4. Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels `a`, `e`, `i`, `o`, `u` appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.

(b) In how many different ways can the letters in the name of the popular 1980's band $BANANARAMA$ be arranged?

(c) In how many different ways can $2n$ students be paired up?

(d) How many different solutions over the natural numbers are there to the following equation?

$$x_1 + x_2 + x_3 + \ldots + x_8 = 100$$

A solution is a specification of the value of each variable x_i. Two solutions are different if different values are specified for some variable x_i.

(e) How many simple graphs are there with n vertices numbered $1, \ldots n$?

Problem 5. How many of the numbers $2, \ldots, n$ are prime? One way to answer this question is to test each number up to n for primality and keep a count. A somewhat more efficient method is to use the "Sieve of Erastosthenes" procedure which you may have learned about in 6.001 (but, don't worry, you needn't know about this). In this problem, we will use the Inclusion-Exclusion Principle to get the count; this approach turns out to be much more efficient when n is large.

Actually, we will use Inclusion-Exclusion to count the number of *composite* (nonprime) integers from 2 to n. Subtracting this from $n - 1$ gives the number of primes.

Let C_n be the set of composites from 2 to n, and let A_m be the set of numbers in the range $m + 1, \ldots, n$ that are divisible by m. Notice that by definition, $A_m = \emptyset$ for $m \geq n$. So

$$C_n = \bigcup_{i=2}^{n-1} A_i.$$

(a) Write C_n in terms of a union of A_p's, where p is prime. Explain why $\sqrt{n}$ is an upper bound on the largest p needed.

(b) What is the cardinality of A_p?

(c) Let P be a set of primes. Give a simple formula for

$$\left| \bigcap_{p \in P} A_p \right|.$$

(d) Use the Inclusion-Exclusion principle to obtain a formula for $|C_{150}|$ in terms of nonempty intersections among the sets $A_2, A_3, A_5, A_7, A_{11}$.

(e) Use this formula to find the number of primes up to 150.

Problem 6. Find the coefficients of

(a) x^{10} in $(x + (1/x))^{100}$

(b) x^k in $(x^2 - (1/x))^n$.

Problem 7. Suppose a generalized World Series between the Sox and the Cardinals involved $2n + 1$ games. As usual, the generalized Series will stop as soon as one team has won more than half the possible games.

(a) Suppose that when the Sox finally win the GSeries, the Cards have managed to win *exactly* r games (so $r \leq n$). How many possible win-loss patterns are possible for the Sox to win the GSeries in this way? Express your answer as a binomial coefficient.

(b) How many possible win-loss patterns are possible for the Sox to win the GSeries when the Cards win *at most* r games? Express your answer as a binomial coefficient.

(c) Give a combinatorial proof that

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}. \tag{1}$$

(d) Verify equation (1) by induction using algebra.

Problem 8. Below is a combinatorial proof of an equation. What is the equation?

Proof. Stinky Peterson owns n newts, t toads, and s slugs. Conveniently, he lives in a dorm with $n+t+s$ other students. (The students are distinguishable, but creatures of the same variety are not distinguishable.) Stinky wants to put one creature in each neighbor's bed. Let W be the set of all ways in which this can be done.

On one hand, he could first determine who gets the slugs. Then, he could decide who among his remaining neighbors has earned a toad. Therefore, $|W|$ is equal to the expression on the left.

On the other hand, Stinky could first decide which people deserve newts and slugs and then, from among those, determine who truly merits a newt. This shows that $|W|$ is equal to the expression on the right.

Since both expressions are equal to $|W|$, they must be equal to each other. $\square$

(Combinatorial proofs are real proofs. They are not only rigorous, but also convey an intuitive understanding that a purely algebraic argument might not reveal. However, combinatorial proofs are usually less colorful than this one.)

Student's Solutions to Problem Set 9

Your name:

Due date: April 25

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.