

Problem Set 7

Due: April 11

Reading:

- [Week 7, pp.19–25](#): Intro. to Number Theory, and
- [Week 8, pp.1–15](#): Sums & Products.

Problem 1. Prove that for any prime, p , and integer, $k \geq 1$,

$$\phi(p^k) = p^k - p^{k-1},$$

where ϕ is Euler's function. *Hint:* Which numbers between 0 and $p^k - 1$ are divisible by p ? How many are there?

Problem 2. Suppose m, n are relatively prime.

(a) Prove that for any a, b , there is an x such that

$$x \equiv a \pmod{m}, \tag{1}$$

$$x \equiv b \pmod{n}. \tag{2}$$

Hint: Congruence (1) holds iff

$$x = jm + a. \tag{3}$$

for some j . So there is such an x only if

$$jm + a \equiv b \pmod{n}. \tag{4}$$

Solve (4) for j .

(b) Prove that there is an x satisfying the congruences (1) and (2) such that $0 \leq x < mn$.

(c) Prove that the x satisfying part (b) is unique.

(d) Conclude from the preceding parts of this problem that

$$\phi(mn) = \phi(m)\phi(n)$$

where ϕ is Euler's function.

Problem 3. Let $S_k = 1^k + 2^k + \dots + (p-1)^k$, where p is an odd prime and k is a positive multiple of $p-1$. Use Fermat's theorem to prove that $S_k \equiv -1 \pmod{p}$.

Problem 4. Find an integer $k > 1$ such that n and n^k agree in the last *two digits* whenever n is a positive number relatively prime to 100.

Problem 5. Is a Harvard degree really worth more than an MIT degree?! Let us say that a person with a Harvard degree starts with \$40,000 and gets a \$20,000 raise every year after graduation, whereas a person with an MIT degree starts with \$30,000, but gets a 20% raise every year. Assume inflation is a fixed 8% every year. That is, \$1.08 a year from now is worth \$1.00 today. (You'll need a calculator to get final answers; if one is not available, it's ok to express the answer as a closed form numerical expression.)

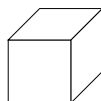
(a) How much is a Harvard degree worth today if the holder will work for n years following graduation?

(b) How much is an MIT degree worth in this case?

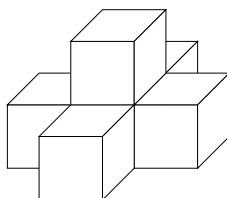
(c) If you plan to retire after twenty years, which degree would be worth more?

Problem 6. Suppose you deposit \$100 into your MIT Credit Union account today, \$99 in one month from now, \$98 in two months from now, and so on. Given that the interest rate is constantly 0.3% per month, how long will it take to save \$5,000?

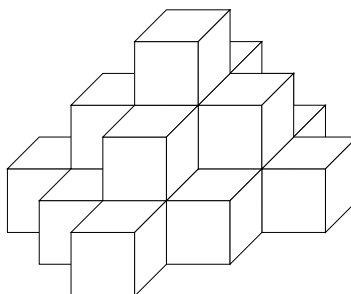
Problem 7. Pharaoh Aha I decides to build a "pyramid" in his honor consisting of a single block:



His successor, Aha II, trumps him by building a larger pyramid:



Not to be outdone, Aha III, builds a still-larger pyramid:



If this continues, how many blocks will Pharoah Aha n require?

Problem 8. Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

Student's Solutions to Problem Set 7

Your name:

Due date: April 11

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.

2. I collaborated on this assignment with:

got help from:¹

and referred to:²

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	