Problem Set 6

Due: April 2

Reading: Notes for

- Week 6.5: Planar Graphs, and
- Week 7, pp.1–18: Intro. to Number Theory.

Problem 1. **(a)** Show that if a planar embedding can be constructed by adding two successive edges to an embedding for a graph, G, then the same planar embedding can be constructed by adding the edges in the reverse order.

(b) Let G be a graph with a planar embedding. Conclude that the same planar embedding can be built up recursively by adding the edges of G in any order.

(c) Conclude that any subgraph of a planar graph is planar.

Problem 2. **(a)** Prove that every planar graph has a vertex of degree at most 5. *Hint:* $e \leq 3v - 6$.

(b) Conclude that every planar graph has width at most 5 and therefore is 6-colorable.[1]
Hint: Use the result of Problem 1(c).

Problem 3. Here is a *very, very fun* game. We start with two distinct, positive integers written on a blackboard. Call them a and b. You and I now take turns. (I'll let you decide who goes first.) On each player's turn, he or she must write a new positive integer on the board that is the difference of two numbers that are already there. If a player can not play, then he or she loses.

For example, suppose that 12 and 15 are on the board initially. Your first play must be 3, which is $15 - 12$. Then I might play 9, which is $12 - 3$. Then you might play 6, which is $15 - 9$. Then I can not play, so I lose.

[1] From Pset 5: A simple graph, G, is said to have ***width***, w, iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most w vertices that precede it in the sequence.

(a) Show that every number on the board at the end of the game is a multiple of $\gcd(a, b)$.

(b) Show that every positive multiple of $\gcd(a, b)$ up to $\max(a, b)$ is on the board at the end of the game.

(c) Describe a strategy that lets you win this game every time.

Problem 4. **(a)** Use the Fundamental Theorem of Arithmetic (that the factorization into primes of an integer greater than 1 is unique) to give simple proofs of a few of the properties of gcd and divisibility listed in Lemma 3.4 of the Number Theory.[2]

Suppose m and n are relatively prime. Use the Fundamental Theorem of Arithmetic (that the factorization into primes of an integer greater than 1 is unique) to give simple proofs of:

(b) $mn \mid a \quad \text{iff} \quad m \mid a$ and $n \mid a$.

(c)

$$\begin{aligned} &x \text{ is relatively prime to } mn \\ &\quad \text{iff} \quad x \text{ is relatively prime to } m \text{ and } x \text{ is relatively prime to } n. \end{aligned}$$

Problem 5. Albert decides to entertain the class with a magic trick. He says:

1. Pick any 5 digit number containing at least two different digits.
2. Shuffle the digits to obtain a different number.
3. Subtract the smaller number from the larger.
4. Now sum the digits of the result.
5. Repeat step 4 until you have only one digit, and write down your answer.

He announces the right answer without seeing the paper, but the 6.042 students are not impressed. This problem demonstrates why they were not impressed with Albert's ”magic.”

[2]These properties were proved in the Notes strictly from the definitions, which took more care. But it would have been “cheating” to prove them in the first place using the Fundamental Theorem, since the proof of the Fundamental Theorem builds on these properties.

(a) Show that taking *any* nonnegative integer (not necessarily a 5-digit number), rearranging its digits to form a new number, and finding the difference between the two numbers, will always result in a multiple of 9.

(b) Show that summing the digits of a positive integer results in an integer that is congruent to it modulo 9.

(c) Show that for any 5 digit number, this procedure always terminates with the same digit. What would happen if the starting number had more than 5 digits?

Problem 6. Suppose that p is a prime and $0 < k < p$.

(a) k is *self-inverse* if $k^2 \equiv 1 \pmod{p}$. Prove that k is self-inverse iff either $k = 1$ or $k = p - 1$.

Hint: $k^2 - 1 = (k - 1)(k + 1)$

(b) Wilson's Theorem asserts

Theorem 6.1 (Wilson's Theorem). *If p is a prime, then*

$$(p - 1)! \equiv -1 \pmod{p}$$

The English mathematician Edward Waring said that this theorem would probably be very difficult to prove because there was no adequate notation for primes. Gauss proved it while standing (on one foot, it is rumored). He suggested that Waring failed for lack of notions, not notations. Prove Wilson's Theorem. *Hint:* While standing on one foot, think about pairing each term in $(p - 1)!$ with its multiplicative inverse.

Student's Solutions to Problem Set 6

Your name:

Due date: April 2

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.