Problem Set 5

Due: March 19

Reading: Notes for Week 6

Problem 1. This problem generalizes the result proved in Week 5 Notes that any graph with maximum degree at most w is $(w+1)$-colorable.

A simple graph, G, is said to have ***width***, w, iff its vertices can be arranged in a sequence such that each vertex is adjacent to at most w vertices that precede it in the sequence. If the degree of every vertex is at most w, then the graph obviously has width at most w —just list the vertices in any order.

(a) Describe an example of a graph with 100 vertices, width 3, but *average* degree more than 5. *Hint:* Don't get stuck on this; if you don't see it after five minutes, ask the staff for a hint.

(b) Prove that every graph with width at most w is $(w+1)$-colorable.

(c) Prove that the average degree of a graph of width w is at most $2w$.

Problem 2. In this problem you will prove:

Theorem. *A graph G is 2-colorable iff it contains no odd length cycle.*

As usual with "iff" assertions, the proof splits into two proofs: part (a) asks you to prove that the left side of the "iff" implies the right side. The other problem parts prove that the right side implies the left.

(a) Assume the left side and prove the right side. Three to five sentences should suffice.

(b) Now assume the right side. As a first step toward proving the left side, explain why we can focus on a single connected component H within G.

(c) As a second step, explain how to 2-color any tree.

(d) Choose any 2-coloring of a spanning tree, T, of H. Prove that H is 2-colorable by showing that any edge *not* in T must also connect different-colored vertices.

Problem 3. Prove that if graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ both have maximum degree 1, then the graph $G = (V, E_1 \cup E_2)$ is 2-colorable.

Problem 4. At the Guiness brewery in the eary 1900's, W. S. Gosset (a chemist) and E. S. Beaven (a "maltster") were working to improve barley. Gosset and Beavan planned to grow several varieties of barley in a field and compare the yields. However, local variations in the fertility of the field might skew the results. Their solution was to divide the field into many small plots and grow each crop in several different places.

Similar thinking led to the use of *Latin squares* in experiment design. A Latin square is an $n \times n$ array of numbers such that each row and each column contains every number from 1 to n. For example, here is a 4×4 Latin square:

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

You can imagine that this array is an agricultural field where each square is a small plot, and the number inside indicates the variety of barley planted there.

There are some nice connections between Latin squares and graph theory.

(a) Construct a graph G_n such that there is a one-to-one correspondence between $n \times n$ Latin squares and valid n-colorings of G_n.

(b) Suppose your teammate wrote down a 3×5 "Latin rectangle":

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

To fill in a possible next line of a complete Latin Square, construct a bipartite graph with vertices numbered 1 to 5 on the left and on the right. Let i and j be adjacent iff column i does not contain j. Find a matching in this graph and use it to fill the 4th row.

(c) Show that filling in the $k+1$st row of a $k \times n$ Latin rectangle is equivalent to finding a matching in some bipartite graph with n vertices on the left and right.

(d) Prove that a matching must exist in this bipartite graph and, consequently, a Latin rectangle can always be extended to a Latin square.

Problem 5. If a and b are distinct nodes of a digraph, then a is said to *cover* b if there is an edge from a to b and there is no other path from a to b. If a covers b, the edge from a to b is called a covering edge.

(a) Show that if two DAG's have the same positive path relation, then they have the same set of covering edges.

(b) For any DAG, D, let $\widehat{D}$ be the subgraph of D consisting of only the covering edges. Show that if D is finite and has no self-loops, then D and $\widehat{D}$ have the same positive path relation, that is $D^+ = \widehat{D}^+$.

(c) Conclude that if D is a finite DAG, then $\widehat{D}$ is the unique DAG with the smallest number of edges among all digraphs with the same positive path relation.

(d) Show that the previous result is not true in general infinite DAG's.

Hint: Consider the DAG for the total order on the rational numbers.

Problem 6. Let B_n denote the butterfly network with $N = 2^n$ inputs and N outputs, as defined in Week 5 Notes. Show that the congestion of B_n is exactly $\sqrt{N}$ when n is even.

Hints:

- For the butterfly network, there is a unique path from each input to each output, so the congestion is the maximum number of messages passing through a vertex for any matching of inputs to outputs.
- If v is a vertex at level i of the butterfly network, there is a path from exactly 2^i input vertices to v and a path from v to exactly 2^{n-i} output vertices.
- At which level of the butterfly network must the congestion be worst? What is the congestion at the node whose binary representation is all 0s at that level of the network?

Student's Solutions to Problem Set 5

Your name:	
Due date:	March 19
Submission date:	
Circle your TA/LA:	Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.