Problem Set 4

Due: March 12

Reading: Notes for Week 5

Problem 1. The Mating Ritual described in Week 5 Notes yields a stable perfect match when there are an equal number of boys and girls.

(a) Suppose there are more girls than boys. Define what a stable matching should mean in this case, and explain why applying the Mating Ritual will yield a stable matching in which every boy is married. Then briefly explain why each boy gets married to his optimal spouse. (You can give complete proofs like those in the Notes, but it's also OK just to explain how the proofs in the Notes should be modified to handle the case with more boys.)

(b) The Notes also described an early and valuable application of the Mating Ritual to assigning medical students to hospital residencies. In this case, each student has a preference ranking of all the hospitals, each hospital has a preference ranking of all the students, and in addition, each hospital has a certain number of resident positions to fill. The number of resident positions to be filled typically differs between hospitals, and the total number to be filled may not equal the number of students.

Carefully define what a *stable assignment* of residents to hospitals should mean in the case that the total number of available positions is at least as large as the number of students. Then explain how to extend the Mating Ritual to find such a stable assignment.

Problem 2. In a set of stable marriages between an equal number of boys and girls, call a person *lucky* if their spouse appears in the top half of their preference list.

Claim. *The Mating Algorithm produces a set of marriages with at least one lucky person.*

To prove the Claim, for each girl, G, define a "rejection count" derived variable, $r(G)$, to be the number of boys she has rejected. Similarly, for each boy, B, define a "rejected count" variable, $r(B)$, to be the number of times he has been rejected by girls.

(a) Define the predicate $L(B)$ meaning "B is a lucky boy," in terms of the final value of $r(B)$.

(b) Suppose that on the final day, the value of $r(G)$, averaged over all the girls, is at *least* half the number of boys. Explain why there must be a lucky girl.

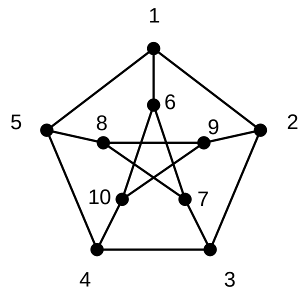
(c) The rejection counts in the Mating Algorithm satisfy an obvious invariant. Use this invariant and the previous problem parts to prove the Claim.

Problem 3. **(a)** Construct a set of marriage preferences in which there are at least three different stable marriage assignments.

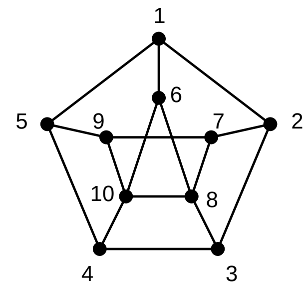
(b) (Optional) Describe how to define a set of marriage preferences among n boys and n girls which have more than $2^{n/4}$ stable assignments.

Problem 4. A property of a graph is said to be *preserved under isomorphism* if whenever G has that property, every graph isomorphic to G also has that property. For example, the property of having five vertices is preserved under isomorphism: if G has five vertices then every graph isomorphic to G also has five vertices.

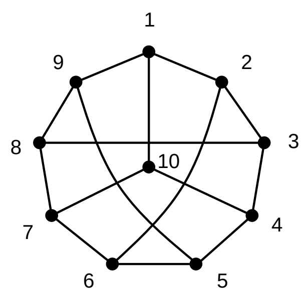
Determine which among the four graphs pictured in Figure 1 are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).



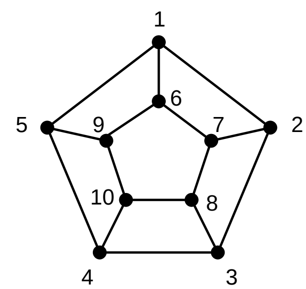
(a) G_1



(b) G_2



(c) G_3



(d) G_4

Figure 1: Which graphs are isomorphic?

Problem 5. Given a simple graph G, we apply the following operation to the graph: pick two vertices $u \neq v$ such that either

1. there is an edge of G between u and v and there is also a path from u to v which does *not* include this edge; in this case, delete the edge $\{u, v\}$.

2. or, there is no path from u to v; in which case, add the edge $\{u, v\}$.

We keep repeating these operations until it is no longer possible to find two vertices $u \neq v$ to which an operation applies.

Assume the vertices of G are the integers $1, 2, \ldots, n$ for some $n \geq 2$. This procedure can be modelled as a state machine whose states are all possible simple graphs with vertices $1, 2, \ldots, n$. The start state is G, and the final states are the graphs on which no operation is possible.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{\{1, 2\}, \{3, 4\}\}$$

What are the possible final states reachable from start state G? Draw them.

(b) For any state, G', let e be the number of edges in G', c be the number of connected components it has, and s be the number of simple cycles. For each of the derived variables below, indicate the *strongest* of the properties that it is guaranteed to satisfy, no matter what the starting graph G is, and briefly explain your answer.

The choices for properties are: *constant*, *strictly increasing*, *strictly decreasing*, *weakly increasing*, *weakly decreasing*, *none of these*. The derived variables are

(i) e

(ii) c

(iii) s

(iv) $e - s$

(v) $c + e$

(vi) $3c + 2e$

(vii) $c + s$

(viii) (c, e), partially ordered coordinatewise (the *product* partial order).

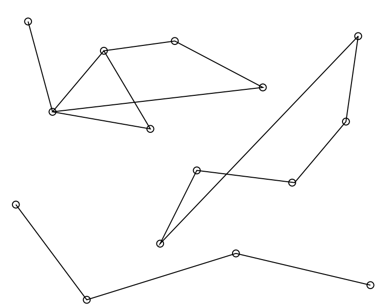
(ix) (c, e), ordered lexicographically

(c) Conclude that the procedure terminates.

(d) Prove that any final state must be a tree on the vertices.

Problem 6. A set, M, of vertices of a graph is a *maximal connected* set if every pair of vertices in the set are connected, and any set of vertices properly containing M will contain two vertices that are not connected.

(a) What are the maximal connected subsets of the following (unconnected) graph?

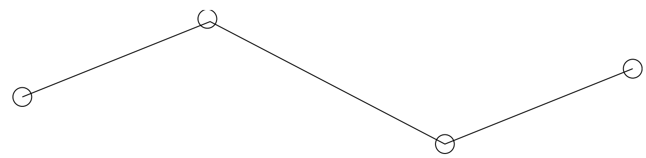


(b) Explain the connection between maximal connected sets and connected components. Prove it.

Problem 7. **(a)** Describe a connected graph such that every vertex is on a simple cycle, but the graph is not 2-connected.

(b) Prove that a graph is 2-connected iff it is connected and every edge is traversed by a simple cycle.

Problem 8. Let's say that a graph has "two ends" if it has exactly two vertices of degree 1 and all its other vertices have degree 2. For example, here is one such graph:



(a) A *line graph* is a graph whose edges can all be traversed by a simple path. So the two-ended graph above is also a line graph of length 4.

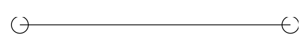
Prove that the following theorem is false by drawing a counterexample.

False Theorem. *Every two-ended graph is a line graph.*

(b) Point out the first erroneous statement in the following alleged proof of the false theorem. Describe the error as best you can.

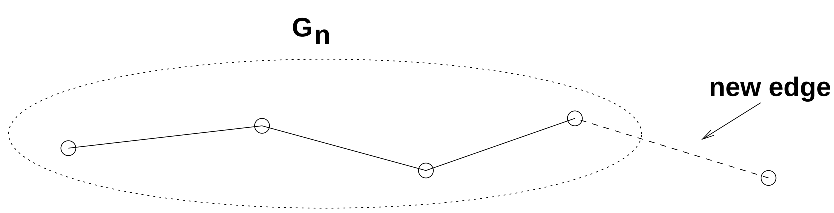
False proof. We use induction. The induction hypothesis is that every two-ended graph with n edges is a path.

Base case ($n = 1$): The only two-ended graph with a single edge consists of two vertices joined by an edge:



Sure enough, this is a line graph.

Inductive case: We assume that the induction hypothesis holds for some $n \geq 1$ and prove that it holds for $n + 1$. Let G_n be any two-ended graph with n edges. By the induction assumption, G_n is a line graph. Now suppose that we create a two-ended graph G_{n+1} by adding one more edge to G_n. This can be done in only one way: the new edge must join an endpoint of G_n to a new vertex; otherwise, G_{n+1} would not be two-ended.



Clearly, G_{n+1} is also a line graph. Therefore, the induction hypothesis holds for all graphs with $n + 1$ edges, which completes the proof by induction.

$\square$

Student's Solutions to Problem Set 4

Your name:

Due date: March 12

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.