Problem Set 2

Due: February 26

Reading: Notes for Week 3

Problem 1. Consider the proper subset partial order, $\subset$, on the power set $\mathcal{P}\{1, 2, \ldots 5\}$.

(a) What is the size of a maximal chain in this partial order? Describe one.

(b) Describe the largest antichain you can find in this partial order.

(c) What are the maximal and minimal elements? Are they maximum and minimum?

(d) Answer the previous part for the $\subset$ partial order on the set $\mathcal{P}\{1, 2, \ldots 5\} - \emptyset$.

Problem 2. Let S be a sequence of n different numbers. A *subsequence* of S is a sequence that can be obtained by deleting elements of S.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of S (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An *increasing subsequence* of S is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of S. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of S.

(a) List all the maximum length increasing subsequences of S, and all the maximum length decreasing subsequences.

Now let A be the *set* of numbers in S. (So $A = \{1, 2, 3, \ldots, 9\}$ for the example above.) There are two straightforward ways to totally order A. The first is to order its elements numerically, that is, to order A with the $<$ relation. The second is to order the elements by which comes first in S; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Next, define the partial order $\prec$ on A defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ and } a <_S a'.$$

(It's not hard to prove that $\prec$ is strict partial order, but you may assume it.)

(b) Draw a diagram of the partial order, $\prec$, on A. What are the maximal elements,... the minimal elements?

(c) Explain the connection between increasing and decreasing subsequences of S, and chains and anti-chains under $\prec$.

(d) Prove that every sequence, S, of length n has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

(e) (Optional, and tricky) Describe a procedure that scans a length n sequence of numbers once from left to right and returns a subsequence of length at least $\sqrt{n}$ that is either increasing or decreasing.

Problem 3. Use induction to prove that the following equation holds for all $n \geq 2$:

$$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n}\right) = \frac{1}{n}$$

Problem 4. A chocolate bar is molded into m rows, with n squares of chocolate in each row. The bar can be split by cutting between rows. For example, if a 8×10 bar is cut between its third and fourth rows, the result would be a two bars, one 3×10 and one 5×10. Similarly, a bar can also be split by cutting between columns. We want to keep making cuts until the bar is completely split into separate squares of chocolate.

Use strong induction to prove that exactly $mn - 1$ cuts are required to split the bar into individual squares.

Problem 5. The following Lemma is true, but the *proof* given for it below is defective. Pinpoint, and illustrate with a counterexample, *exactly* where the proof goes wrong.

Lemma 5.1. *For any prime p and positive integers $n, x_1, x_2, \ldots, x_n$, if $p \mid x_1 x_2 \ldots x_n$, then $p \mid x_i$ for some $1 \leq i \leq n$.*

False proof. Proof by strong induction on n.

Base case $n = 1$: When $n = 1$, we have $p \mid x_1$, therefore we can let $i = 1$ and conclude $p \mid x_i$.

Induction step: Now assuming the claim holds for all $k \leq n$, we must prove it for $n + 1$.

So suppose $p \mid x_1 x_2 \ldots x_{n+1}$. Let $y_n = x_n x_{n+1}$, so $x_1 x_2 \ldots x_{n+1} = x_1 x_2 \ldots x_{n-1} y_n$. Since the righthand side of this equality is a product of n terms, we have by induction that p divides one of them. If $p \mid x_i$ for some $i < n$, then we have the desired i. Otherwise $p \mid y_n$. But since y_n is a product of the two terms x_n, x_{n+1}, we have by strong induction that p divides one of them. So in this case $p \mid x_i$ for $i = n$ or $i = n + 1$. □

Problem 6. While 6.042 was running in the TEAL room, the staff considered having students work in teams of exactly 4 or exactly 7 students. But TEAL tables hold up to nine students, so this plan was abandoned when it was realized then that no full table of nine could be divided into teams.

(a) What is the smallest number, k, such that a group of k or more students can be split into teams of 4 and 7? Prove that $k - 1$ students *cannot* be split in this way.

(b) Now use strong induction to complete the proof of your answer to part (a). Namely, prove by strong induction that all tutorials of k or more students *can* be split into teams of size 4 and 7.

Problem 7. Notes 3 proved that the stacking game with n blocks always ended with the same score.

Define the ***potential***, $p(S)$, of a stack, S, of blocks to be $k(k+1)/2$ where k is the number of blocks in S. Define the potential, $p(A)$, of a set, A, of stacks to be the sum of the potentials of the stacks in A.

Generalize the result in Notes 3 by showing that for any set, A, of stacks, if a sequence of moves starting with A leads to another set, B, of stacks, then the score for this sequence of moves is $p(A) - p(B)$.

Hint: Prove the

Lemma. *If B is the result of* one *move (stack split) from A, then the score for making this move is $p(A) - p(B)$.*

Problem 8. Consider the following equivalent way of viewing the subset take-away game from the Week 2, Friday class problem: for a fixed, finite set, A, let $\mathcal{S}$ initially be all the nonempty proper subsets of A. Players alternately choose a set $B \in \mathcal{S}$ and remove B and all sets that contain B from $\mathcal{S}$; they then continue playing on the updated $\mathcal{S}$. The player who chooses the last set in $\mathcal{S}$ wins.

Use the Well Ordering Property to show that, in any game, one of the players must have a winning strategy. *Hint:* Consider games whose initial set, $\mathcal{S}$, is an arbitrary collection of subsets of A, not necessarily all the proper subsets of A. Reach a contradiction by considering a minimum size game with no winning strategy for either player. What is a useful measure of size of a game?

Student's Solutions to Problem Set 2

Your name:

Due date: February 26

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.