Problem Set 11

Due: May 11

Reading:

- Week 12: Introduction to Probability
- Week 13: Random Variables: Distributions & Sampling

Do any four out of five problems for full credit on this problem set.

Problem 1. Outside of their hum-drum duties as 6.042 LAs, Jeff is trying to learn to levitate using only intense concentration and Jessica is trying to become the world champion flaming torch juggler. Suppose that Jeff's probability of success is $1/6$, Jessica's chance of success is $1/4$, and these two events are independent.

(a) If at least one of them succeeds, what is the probability that Jeff learns to levitate?

(b) If at most one of them succeeds, what is the probability that Jessica becomes the world flaming torch juggler champion?

(c) If exactly one of them succeeds, what is the probability that it is Jeff?

Problem 2. Independently flip three fair coins ("fair" means equally likely to come up with a head or a tail), and let H_i be the indicator variable for a head occurring on the ith flip, for $i = 1, 2, 3$. Define $C ::= H_1 + H_2 + H_3$ to be the number of heads flipped, M to be the indicator variable for the event $[H_1 = H_2 = H_3]$ that all three coins match, and S be the indicator variable for the event $[C \equiv 1 \mod 2]$ that an odd number of heads are flipped.

Class Problem 12F, Prob 3 shows that none of these variables is independent of C, and that the variables H_1, H_2, H_3, S are 3-wise independent, but not mutually independent.

(a) Verify that H_1, S and M are not mutually independent.

(b) Verify that the five variables other than C are pairwise independent.

Since 3-wise independence by definition implies pairwise independence, any two of H_1, H_2, H_3, S are pairwise independent, so we need only verify that H_i and M are pairwise independent and that S and M are pairwise independent.

(c) Verify that no set of three variables including both M and H_i for any $i \in \{1, 2, 3\}$ is 3-wise independent.

Problem 3. Suppose you have three cards: $A\heartsuit$, $A\spadesuit$, and a Jack. From these, you choose a random hand (that is, each card is equally likely to be chosen) of two cards, and let K be the number of Aces in your hand. You then randomly pick one of the cards in the hand and reveal it.

(a) Describe a simple probability space (that is, outcomes and their probabilities) for this scenario, and list the outcomes in each of the following events:

1. $[K \geq 1]$, (that is, your hand has an Ace in it),
2. $A\heartsuit$ is in your hand,
3. the revealed card is an $A\heartsuit$,
4. the revealed card is an Ace.

(b) Then calculate $\Pr\{K = 2 \mid E\}$ for E equal to each of the four events in part (a). Notice that most, but *not all*, of these probabilities are equal.

Now suppose you have a deck with d distinct cards, a different kinds of Aces (including an $A\heartsuit$), you draw a random hand with h cards, and then reveal a random card from your hand.

(c) Prove that $\Pr\{A\heartsuit \text{ is in your hand}\} = h/d$.

(d) Prove that

$$\Pr\{K = 2 \mid A\heartsuit \text{ is in your hand}\} = \Pr\{K = 2\} \cdot \frac{2d}{ah}. \tag{1}$$

(e) Conclude that $\Pr\{K = 2 \mid \text{the revealed card is an Ace}\} = \Pr\{K = 2 \mid A\heartsuit \text{ is in your hand}\}$.

Problem 4. Let's play a game! We repeatedly flip a fair coin. You have the sequence HHT, and I have the sequence HTT. If your sequence comes up first, then you win. If my sequence comes up first, then I win. For example, if the sequence of tosses is:

$$TTHTHT\underline{HHT}$$

then you win. What is the probability that you win? It may come as a surprise that the answer is very different from 1/2.

This problem is tricky, because the game could go on for an arbitrarily long time. Draw enough of the tree diagram to see a pattern, and then sum up the probabilities of the (infinitely many) outcomes in which you win.

It turns out that for any sequence of three flips, there is another sequence that is likely to come up before it. So there is no sequence which turns up earliest! ...and given any sequence, knowing how to pick another sequence that comes up sooner more than half the time gives you a nice chance to fool people gambling at a bar :-)

Problem 5. The Boston Globe, April 25, 2007 reported that the election between two candidates for Natick Town Moderator resulted in a tie after 4000 residents voted. The article went on to say

> Just what are the odds of a tie in a townwide election? Several election specialists said they could not think of one occurring previously in the state.
>
> "I'd suspect it's not astronomical in the cosmic sense, but it certainly is in the earthly sense," said Thomas Patterson, an elections specialist at Harvard's Kennedy School of Government.

(a) Suppose that each of the 4000 residents was equally likely to vote for either of the two candidates. Estimate the probability of a tie.

(b) Comment on the plausibility of the "random vote" model of part (a). Can you suggest other models that seem more convincing?

Student's Solutions to Problem Set 11

Your name:

Due date: May 11

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '07 course materials.