Problem Set 1

Due: February 20

Reading: Notes for Weeks 1 and 2

Problem 1. Show that $\log_7 n$ is either an integer or irrational, where n is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts. (This problem will be graded on the clarity and simplicity of your proof. If you can't figure out how to prove it, ask the staff for help and they'll tell you how.)

Problem 2. There are two kinds of people in the land of Paradox: liars and truth-tellers. As you might expect, liars claim a statement is true iff it is actually false, while truth-tellers do the opposite. There is only one direction to go to get out of the land of Paradox, either North or South. Bill is a resident of Paradox and you say to him "$L \oplus N$," where L means "You, Bill, are a liar" and N means "the direction out of Paradox is North." Draw a truth table showing that North is the direction out of Paradox iff Bill claims what you told him is true. (Remember, you start off with no idea whether Bill is a liar or truth-teller.)

Problem 3. Describe a simple recursive procedure which, given a positive integer argument, n, produces a truth table whose rows are all the assignments of truth values to n propositional variables. For example, for $n = 2$, the table might look like:

T	**T**
T	**F**
F	**T**
F	**F**

Your description can be in English, or a simple program in some familiar language (say Scheme or Java), but if you do write a program, be sure to include some sample output.

Problem 4. Translate the following sentence into a predicate formula:

> There is a student who has emailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are

- equality, and
- $E(x, y)$, meaning that "x has sent e-mail to y."

Problem 5. Express each of the following predicates and propositions in formal logic notation. The domain of discourse is the nonnegative integers, $\mathbb{N}$. Moreover, in addition to the propositional operators, variables and quantifiers, you may define predicates using addition, multiplication, and equality symbols, but no *constants* (like 0, 1,...) and no *exponentiation* (like x^y). For example, the proposition "n is an even number" could be written

$$\exists m. \ (m + m = n).$$

(a) n is the sum of two fourth-powers (a fourth-power is k^4 for some integer k).

Since the constant 0 is not allowed to appear explicitly, the predicate "$x = 0$" can't be written directly, but note that it could be expressed in a simple way as:

$$x + x = x.$$

Then the predicate $x > y$ could be expressed

$$\exists w. \ (y + w = x) \wedge (w \neq 0).$$

Note that we've used "$w \neq 0$" in this formula, even though it's technically not allowed. But since "$w \neq 0$" is equivalent to the allowed formula "$\neg(w + w = w)$," we can use "$w \neq 0$" with the understanding that it abbreviates the real thing. And now that we've shown how to express "$x > y$," it's ok to use it too.

(b) $x = 1$.

(c) m is a divisor of n (notation: $m \mid n$)

(d) n is a prime number (hint: use the predicates from the previous parts)

(e) n is a power of 3.

Problem 6. Prove that
$$[\forall x.\ \neg P(x)] \longrightarrow \neg\exists z.\ P(z).$$
is valid. (Use the validity proof from lecture 2W and from the subsection on Validity in Week 2 Notes as guides to writing your proof.)

Problem 7. Let A, B, and C be sets. Prove that:
$$A \cup B \cup C = (A - B) \cup (B - C) \cup (C - A) \cup (A \cap B \cap C).$$
You are welcome to use a diagram to aid your own reasoning, but your proof must be text.

Problem 8. **(a)** Give an example where the following result fails:

False Theorem. *For sets A, B, C, and D, let*
$$L ::= (A \cup C) \times (B \cup D),$$
$$R ::= (A \times B) \cup (C \times D).$$
Then $L = R$.

(b) Identify the mistake in the following proof of the False Theorem.

Bogus proof. Since L and R are both sets of pairs, it's sufficient to prove that $(x, y) \in L \longleftrightarrow (x, y) \in R$ for all x, y.

The proof will be a chain of iff implications:

	$(x, y) \in L$
iff	$x \in A \cup C$ and $y \in B \cup D$
iff	either $x \in A$ or $x \in C$, and either $y \in B$ or $y \in D$
iff	($x \in A$ and $y \in B$) or else ($x \in C$ and $y \in D$)
iff	$(x, y) \in A \times B$, or $(x, y) \in C \times D$
iff	$(x, y) \in (A \times B) \cup (C \times D)$
iff	$(x, y) \in R$.

$\square$

(c) Fix the proof to show that $R \subseteq L$.

Problem 9. If a set, A, is finite, then $|A| < 2^{|A|} = |\mathcal{P}(A)|$, and so there is no bijection from $\mathcal{P}(A)$ to A. Show that this is still true if A is infinite. *Hint:* Remember Russell's paradox and consider

$$\{f(B) \in A \mid B \subseteq A \text{ and } f(B) \notin B\}$$

where f is such a bijection.

Student's Solutions to Problem Set 1

Your name:

Due date: February 20

Submission date:

Circle your TA/LA: Chiyoun Jay Jeffrey Jessica Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total	

[1] People other than course staff.
[2] Give citations to texts and material other than the Spring '07 course materials.