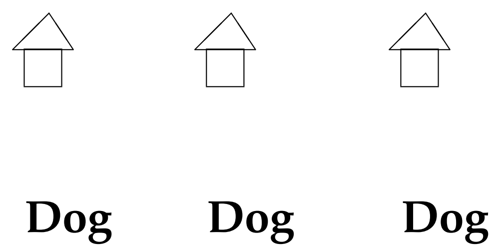
Planar Graphs

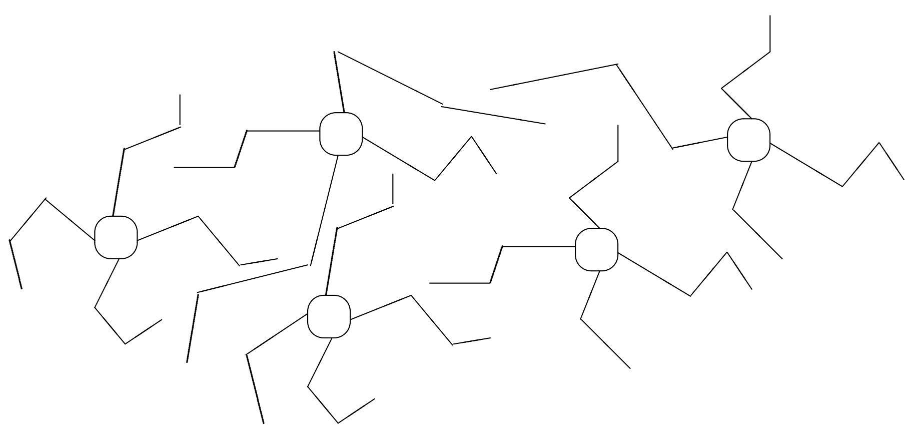
1 Drawing Graphs in the Plane

Here are three dogs and three houses.



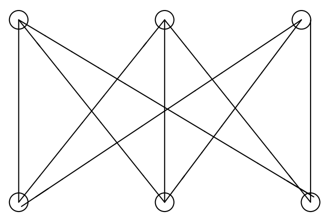
Can you find a path from each dog to each house such that no two paths intersect?

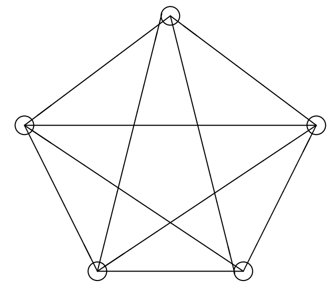
A *quadapus* is a little-known animal similar to an octopus, but with four arms. Here are five quadapi resting on the seafloor:



Can each quadapus simultaneously shake hands with every other in such a way that no arms cross?

Informally, a ***planar graph*** is a graph that can be drawn in the plane so that no edges cross, as in a map of showing the borders of countries or states. Thus, these two puzzles are asking whether the graphs below are planar; that is, whether they can be redrawn so that no edges cross. The first graph is called the ***complete bipartite graph***, $K_{3,3}$, and the second is K_5.





In each case, the answer is, "No— but almost!" In fact, each drawing *would* be possible if any single edge were removed.

Planar graphs have applications in circuit layout and are helpful in displaying graphical data, for example, program flow charts, organizational charts, and scheduling conflicts. We will use them to prove a wonderful mathematical fact that was first proved by the ancient Greeks.

When wires are arranged on a surface, like a circuit board or microchip, crossings require troublesome three-dimensional structures. When Steve Wozniak designed the disk drive for the early Apple II computer, he struggled mightily to achieve a nearly planar design:

> For two weeks, he worked late each night to make a satisfactory design. When he was finished, he found that if he moved a connector he could cut down on feedthroughs, making the board more reliable. To make that move, however, he had to start over in his design. This time it only took twenty hours. He then saw another feedthrough that could be eliminated, and again started over on his design. "The final design was generally recognized by computer engineers as brilliant and was by engineering aesthetics beautiful. Woz later said, 'It's something you can only do if you're the engineer and the PC board layout person yourself. That was an artistic layout. The board has virtually no feedthroughs.'"[a]

[a] From apple2history.org which in turn quotes *Fire in the Valley* by Freiberger and Swaine.

2 Continuous & Discrete Faces

Planar graphs are graphs that can be drawn in the plane —like familiar maps of countries or states. "Drawing" the graph means that each vertex of the graph corresponds to a distinct point in the plane, and if two vertices are adjacent, their vertices are connected by a smooth, non-self-intersecting curve. None of the curves may "cross" —the only points that may appear on more than one curve are the vertex points. These curves are the boundaries of connected regions of the plane called the *continuous faces* of the drawing.

For example, the drawing in Figure 1 has four continuous faces. Face IV, which extends off to infinity in all directions, is called the ***outside face***.

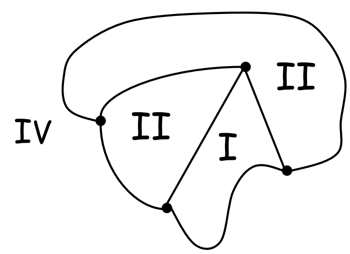


Figure 1: A Planar Drawing with Four Faces.

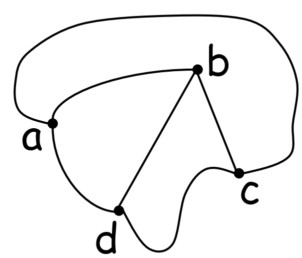


Figure 2: The Drawing with Labelled Vertices.

This definition of planar graphs is perfectly precise, but completely unsatisfying: it invokes smooth curves and continuous regions of the plane to define a property of a discrete data type. So the first thing we'd like to find is a discrete data type that represents planar drawings.

The clue to how to do this is to notice that the vertices along the boundary of each of the faces in Figure 1 form a simple cycle. For example, labeling the vertices as in Figure 2, the simple cycles for the face boundaries are

`abca` `abda` `bcdb` `acda`.

Since every edge in the drawing appears on the boundaries of exactly two continuous faces, every edge of the simple graph appears on exactly two of the simple cycles.

Vertices around the boundaries of states and countries in an ordinary map are always simple cycles, but oceans are slightly messier. The ocean boundary is the set of all boundaries of islands and continents in the ocean; it is a *set* of simple cycles (this can happen for countries too —like Bangladesh). But this happens because islands (and the two parts of Bangladesh) are not connected to each other. So we can dispose of this complication by treating each connected component separately.

But general planar graphs, even when they are connected, may be a bit more complicated than maps. For example a planar graph may have a "bridge," as in Figure 3. Now the cycle around the

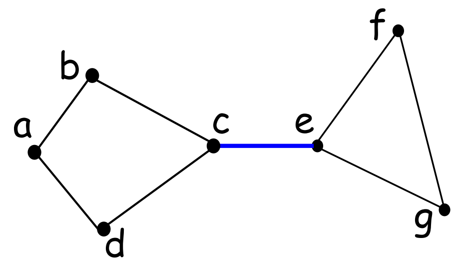


Figure 3: A Planar Drawing with a *Bridge*.

outer face is

`abcefgecda`.

This is not a simple cycle, since it has to traverse the bridge c—e twice.

Planar graphs may also have "dongles," as in Figure 4. Now the cycle around the inner face is

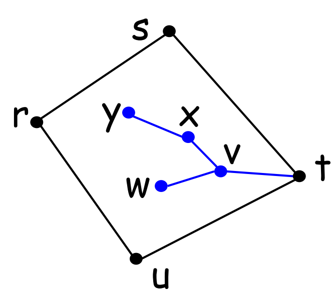


Figure 4: A Planar Drawing with a *Dongle*.

`rstvxyxvwvtur`,

because it has to traverse *every* edge of the dongle twice —once "coming" and once "going."

But bridges and dongles are really the only complications, which leads us to the discrete data type of *planar embeddings* that we can use in place of continuous planar drawings. Namely, we'll define a planar embedding recursively to be the set of boundary-tracing cycles we could get drawing one edge after another.

3 Planar Embeddings

By thinking of the process of drawing a planar graph edge by edge, we can give a useful recursive definition of planar embeddings.

Definition 3.1. A *planar embedding* of a *connected* graph consists of a nonempty set of cycles of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

- **Base case:** If G is a graph consisting of a single vertex, v, then a planar embedding of G has one discrete face, namely the length zero cycle, v.

- **Constructor Case:** (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ, of the planar embedding. That is, γ is a cycle of the form

$$a \dots b \cdots a.$$

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with the same discrete faces as G, except that face γ is replaced by the two discrete faces[1]

$$a \dots ba \quad \text{and } ab \cdots a,$$

as illustrated in Figure 5.

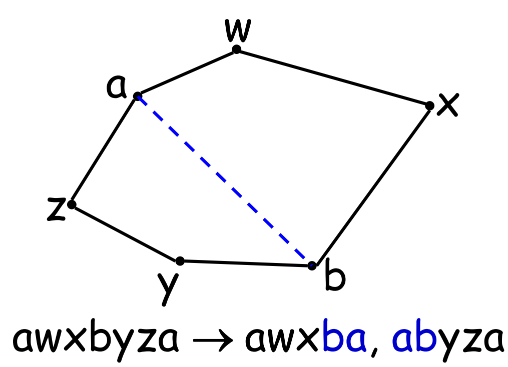


Figure 5: The Split a Face Case.

- **Constructor Case:** (add a bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let a be a vertex on a discrete face, γ, in the embedding of G. That is, γ is of the form

$$a \dots a.$$

Similarly, let b be a vertex on a discrete face, δ, in the embedding of H, so δ is of the form

$$b \cdots b.$$

[1] There is one exception to this rule. If G is a line graph beginning with a and ending with b, then the cycles into which γ splits are actually the same. That's because adding edge a—b creates a simple cycle graph, C_n, that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces. But since this is the only situation in which two faces are actually the same cycle, this exception is better explained in a footnote than mentioned explicitly in the definition.

Then the graph obtained by connecting G and H with a new edge, a—b, has a planar embedding whose discrete faces are the union of the discrete faces of G and H, except that faces γ and δ are replaced by one new face

$$a \ldots ab \cdots ba.$$

This is illustrated in Figure 6, where the faces of G and H are:

$$G : \{\texttt{axyza, axya, ayza}\} \qquad H : \{\texttt{btuvwb, btvwb, tuvt}\},$$

and after adding the bridge `a—b`, there is a single connected graph with faces

$$\{\texttt{axyzabtuvwba, axya, ayza, btvwb, tuvt}\}.$$

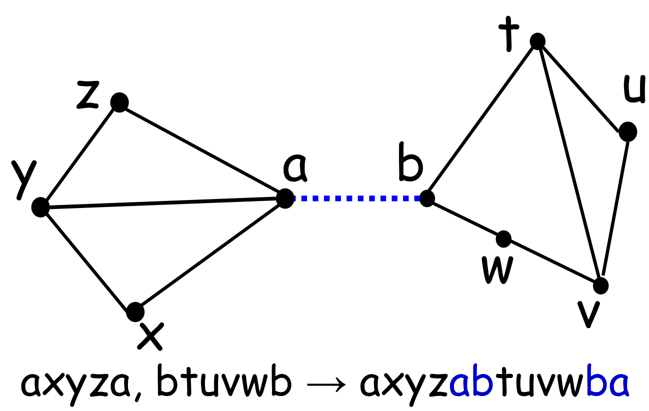


Figure 6: The Add Bridge Case.

An arbitrary graph is *planar* iff each of its connected components has a planar embedding.

3.1 What outer face?

Notice that the definition of planar embedding does not distinguish an "outer" face. There really isn't any need to distinguish one.

In fact, a planar embedding could be drawn with any given face on the outside. An intuitive explanation of this is to think of drawing the embedding on a *sphere* instead of the plane. Then any face can be made the outside face by "puncturing" that face of the sphere, stretching the puncture hole to an circle around the rest of the faces, and flattening the circular drawing onto the plane.

So pictures that show different "outside" boundaries may actually be illustrations of the same planar embedding.

This is what justifies the "add bridge" case in a planar embedding: whatever face is chosen in the embeddings of each of the disjoint planar graphs, we can draw a bridge between them without needing to cross any other edges in the drawing, because we can assume the bridge connects two "outer" faces.

4 Euler's Formula

The value of the recursive definition is that it provides a powerful technique for proving properties of planar graphs, namely, structural induction.

One of the most basic properties of a connected planar graph is that its number of vertices and edges determines the number of faces in every possible planar embedding:

Theorem 4.1 (Euler's Formula)**.** *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

For example, in Figure 1, $|V| = 4$, $|E| = 6$, and $f = 4$. Sure enough, $4 - 6 + 4 = 2$, as Euler's Formula claims.

Proof. The proof is by structural induction on the definition of planar embeddings. Let $P(\mathcal{E})$ be the proposition that $v - e + f = 2$ for an embedding, $\mathcal{E}$.

Base case: ($\mathcal{E}$ is the one vertex planar embedding). By definition, $v = 1$, $e = 0$, and $f = 1$, so $P(\mathcal{E})$ indeed holds.

Constructor case: (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, $\gamma = a \dots b \cdots a$, of the planar embedding.

Then the graph obtained by adding the edge a—b to the edges of G has a planar embedding with one more face and one more edge than G. So the quantity $v - e + f$ will remain the same for both graphs, and since by structural induction this quantity is 2 for G's embedding, it's also 2 for the embedding of G with the added edge. So P holds for the constructed embedding.

Constructor case: (add bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Then connecting these two graphs with a cross yields a planar embedding of a connected graph with $v_G + v_H$ vertices, $e_G + e_H + 1$ edges, and $f_G + f_H - 1$ faces. But

$$\begin{aligned}
&(v_G + v_H) - (e_G + e_H + 1) + (f_G + f_H - 1) \\
&= (v_G - e_G + f_G) + (v_H - e_H + f_H) - 2 \\
&= (2) + (2) - 2 && \text{(by structural induction hypothesis)} \\
&= 2.
\end{aligned}$$

So $v - e + f$ remains equal to 2 for the constructed embedding. That is, P also holds in this case.

This completes the proof of the constructor cases, and the theorem follows by structural induction.

□

4.1 Number of Edges versus Vertices

Like Euler's formula, the following lemmas follow by structural induction directly from the definition of planar embedding.

Lemma 4.2. *In a planar embedding of a connected graph, each edge is traversed once by each of two different faces, or is traversed exactly twice by one face.*

Lemma 4.3. *In a planar embedding of a graph with at least three vertices, each face is of length at least three.*

Corollary 4.4. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 4.2, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 4.3, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{1}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (1) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

□

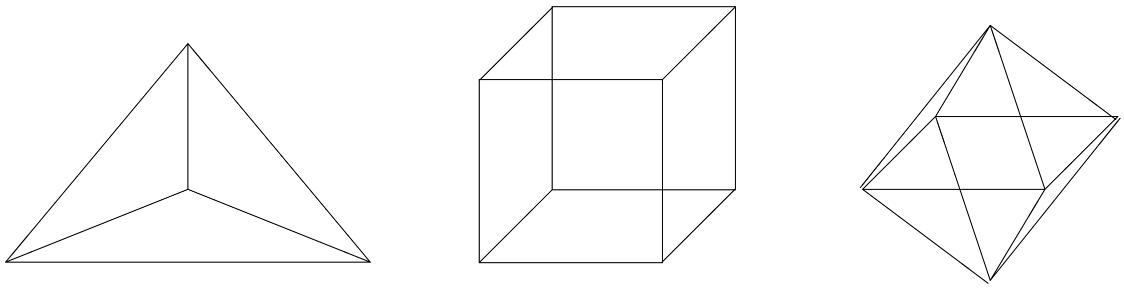
Corollary 4.4 lets us prove that the quadapi can't all shake hands without crossing. Representing quadapi by vertices and the necessary handshakes by edges, we get the complete graph, K_5. Shaking hands without crossing amounts to showing that K_5 is planar. But K_5 is connected, has 5 vertices and 10 edges, and $10 > 3 \cdot 5 - 6$. This violates the condition of Corollary 4.4 required for K_5 to be planar.

Corollary 4.4 also leads to an easy proof that every planar graph is 6-colorable, as shown in Problem Set 6. Building on this, it's not too hard to prove that every planar graph is 5-colorable, but we won't go into it. The proof that every planar graph is 4-colorable is one of the celebrated Mathematical results of the latter half of the Twentieth Century. Even after three decades of simplification, it requires graduate-level training and computer support to follow the proof.

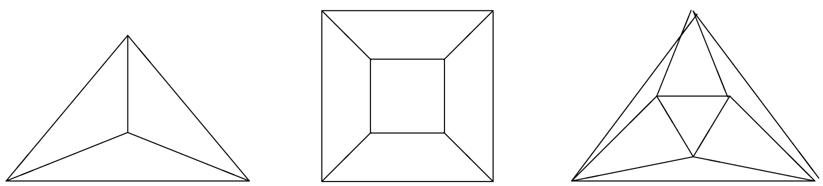
5 Classifying Polyhedra

The Pythagoreans had two great mathematical secrets, the irrationality of 2 and a geometric construct that we're about to rediscover!

A ***polyhedron*** is a convex, three-dimensional region bounded by a finite number of polygonal faces. If the faces are identical regular polygons and an equal number of polygons meet at each corner, then the polyhedron is ***regular***. Three examples of regular polyhedra are shown below: the tetrahedron, the cube, and the octahedron.



How many more polyhedra are there? Imagine putting your eye very close to one face of a translucent polyhedron. The edges of that face would ring the periphery of your vision and all other edges would be visible within. For example, the three polyhedra above would look something like this:



Thus, we can regard the corners and edges of these polyhedra as the vertices and edges of a connected planar graph. (This is another logical leap based on geometric intuition.) This means Euler's formula for planar graphs can help guide our search for regular polyhedra.

Let m be the number of faces that meet at each corner of a polyhedron, and let n be the number of sides on each face. In the corresponding planar graph, there are m edges incident to each of the v vertices. Since each edge is incident to two vertices, we know:

$$mv = 2e$$

Also, each face is bounded by n edges. Since each edge is on the boundary of two faces, we have:

$$nf = 2e$$

Solving for v and f in these equations and then substituting into Euler's formula gives:

$$\frac{2e}{m} - e + \frac{2e}{n} = 2$$
$$\frac{1}{m} + \frac{1}{n} = \frac{1}{e} + \frac{1}{2}$$

The last equation places strong restrictions on the structure of a polyhedron. Every nondegenerate polygon has at least 3 sides, so $n \geq 3$. And at least 3 polygons must meet to form a corner, so $m \geq 3$. On the other hand, if either n or m were 6 or more, then the left side of the equation could be at most $1/3 + 1/6 = 1/2$, which is less than the right side. Checking the finitely-many cases that remain turns up only five solutions. For each valid combination of n and m, we can compute the associated number of vertices v, edges e, and faces f. And polyhedra with these properties do actually exist:

n	m	v	e	f	polyhedron
3	3	4	6	4	tetrahedron
4	3	8	12	6	cube
3	4	6	12	8	octahedron
3	5	12	30	20	icosahedron
5	3	20	30	12	dodecahedron

The last polyhedron in this list, the dodecahedron, was the other great mathematical secret of the Pythagorean sect. These five, then, are the only possible regular polyhedra.

So if you want to put more than 20 geocentric satellites in orbit so that they *uniformly* blanket the globe —tough luck!