Massachusetts Institute of Technology
6.042J/18.062J, Spring '07: Mathematics for Computer Science
Prof. Albert R. Meyer
revised April 13, 2007, 159 minutes

Mini-Quiz Apr. 13

Your name:_____

Circle the name of your TA/LA:

Chiyoun Jay Jeffrey Jessica Tina

- This quiz is **closed book**. Total time is 25 minutes.
- There are five (5) problems totalling 15 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	3		
2	3		
3	3		
4	3		
5	3		
Total	15		

Problem 1 (3 points)**.**

(a) (1 point) Calculate the value of $\phi(100)$.

(b) (2 points) Assume n is a positive integer greater than 9, and relatively prime to 100. Explain why the last two digits of n and n^{121} are the same.

Problem 2 (3 points)**.** Use the fact that

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

to show that

$$\sum_{i=1}^{\infty} i x^i = \frac{x}{(1-x)^2}.$$

for $|x| < 1$.

Problem 3 (3 points)**.** Let's say you earn $20,000 immediately and get a $10,000 raise every year after that. Assume that the interest rate is a fixed 10% every year, that is, $11 a year from now is worth $10 today. If you can work forever, how much is the total salary worth in today's dollars?

(a) (1 point) Write a series summation to express this quantity.

(b) (2 points) What is the quantity? You may write a simple arithmetic expression (that is, no indexed sums or products) for its value **or** you may simply give its numerical value. (Do *not* use a calculator.)

Hint: Problem 2 above.

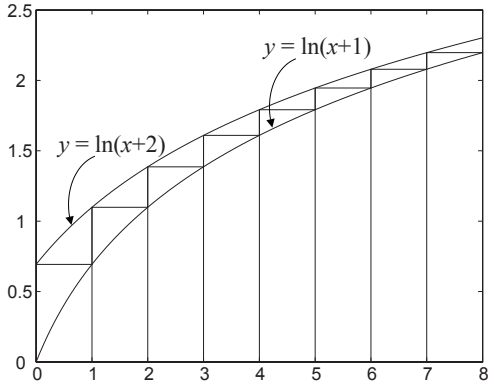
Problem 4 (3 points)**.** There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 mm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson *stretches* the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here's what happens in the first few seconds:

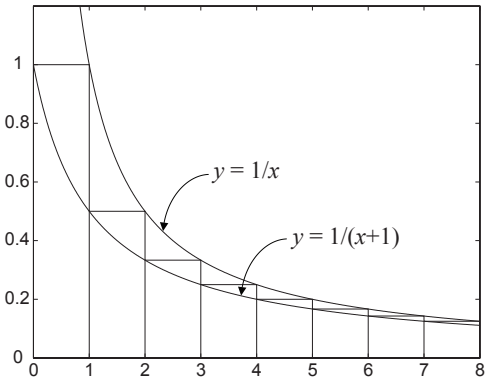
- The bug walks 1 mm in the first second, so 999 mm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 mm behind the bug and 1998 mm ahead.
- The bug walks another 1 mm in the next second, leaving 3 mm behind and 1997 mm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ mm behind the bug and $1997 \cdot (3/2) = 2995.5$ mm ahead.
- The bug walks another 1 mm in the third second, and so on.

(a) (1 point) Over the first n seconds, what fraction of the rug does the bug cross altogether? Express your answer in terms of the Harmonic number H_n.

(b) (2 points) Can the bug cross the entire rug? Briefly explain why.

Problem 5 (3 points)**.** Circle all the correct inequalities below. Assume n is an integer larger than 1. Do *not* use a calculator. *Hint:* You may find the graphs helpful.





- $\sum_{i=1}^{n} \ln(i+1) \geq \ln 6 + \int_{2}^{n} \ln(x+1)dx$
- $\sum_{i=1}^{n} \ln(i+1) \leq \int_{0}^{n} \ln(x+2)dx$
- $\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_{1}^{n} \ln(x+1)dx$
- $\sum_{i=1}^{n} \frac{1}{i} \geq \int_{0}^{n} \frac{1}{x+1}dx$
- $\sum_{i=1}^{n} \frac{1}{i} \leq 1.5 + \int_{3}^{n} \frac{1}{x}dx$
- $\sum_{i=1}^{n} \frac{1}{i} \geq 1 + \int_{1}^{n} \frac{1}{x}dx$

1 Appendix

Definition. The value of *Euler's totient function*, $\phi(n)$, is defined to be the number of positive integers less than n that are relatively prime to n.

Lemma (Euler Totient Function Equations)**.**

$$\phi(p^k) = p^k - p^{k-1} \qquad \textit{for prime, } p\textit{, and } k > 0,$$
$$\phi(mn) = \phi(m) \cdot \phi(n) \qquad \textit{when } \gcd(m, n) = 1.$$

Theorem (Euler's Theorem)**.** *If k and n are relatively prime, then*

$$k^{\phi(n)} \equiv 1 \pmod{n}$$

Definition. The n-th harmonic number is defined as $H_n ::= \sum_{i=1}^{n} \frac{1}{i}$.

$$\ln(n+1) \leq H_n \leq 1 + \ln n$$