Mini-Quiz Mar. 21

Your name: ______________________________

Circle the name of your TA/LA:

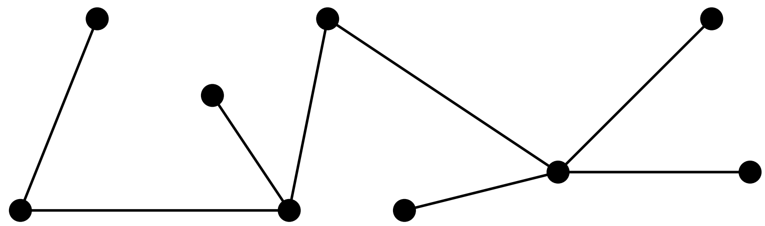
Chiyoun Jay Jeffrey Jessica Tina

- This quiz is **closed book**. Total time is 25 minutes.
- There are four (4) problems totalling 15 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

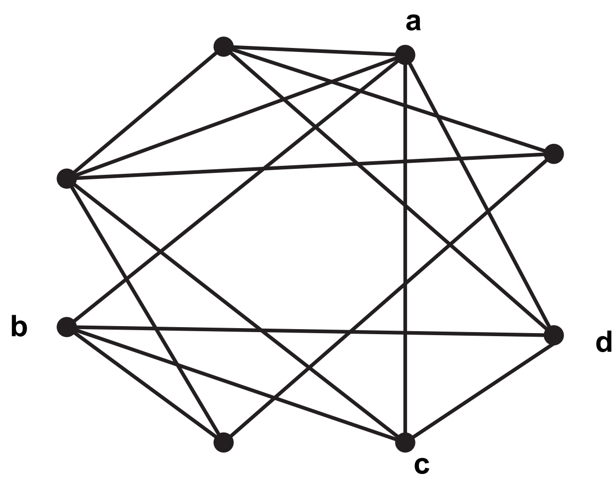
Problem	Points	Grade	Grader
1	4		
2	3		
3	4		
4	4		
Total	15		

Problem 1 (4 points). **(a) (2 points)**



1. Provide a valid coloring of this graph, by labelling each of the vertices with a color. Use as few colors as possible.
2. Explain why the chromatic number of the graph is at least 2.

(b) (2 points)



1. Provide a valid coloring of this graph, by labelling each of the vertices with a color. Use as few colors as possible.
2. Explain why the chromatic number of the graph is at least 4.

Problem 2 (3 points)**.** The team of LA's and TA's think they are being overworked and have decided to oust Albert, and teach their own recitations. They will offer five separate recitations, each at different time slots. Each recitation is only allowed to have up to 20 students.

Each student only has some subset of schedule time slots open.

The LA's and TA's want to determine whether a chosen set of five recitation time slots will allow each student to be assigned to a recitation they can attend, and, if so, which recitation each student should be assigned to.

(a) (1 point) What should we use to model this problem? Circle the best option below:

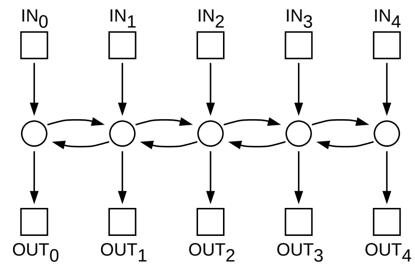
- Stable matching
- Vertex coloring
- Bipartite matching

(b) (2 points) Describe how to model the problem, based on your choice above. Be sure to describe whatever may be relevant: preference lists, vertices, edges, or partitions, for example, *as well as* a brief description of how to interpret the output stable matching/ vertex coloring/ bipartite matching. (This is a *modeling problem*; we are not looking for conditions under which solutions exist nor descriptions of algorithms that solve the problems in part (a).)

Problem 3 (4 points)**.** Prove that every degree-constrained bipartite graph, G, with vertex partition L, R has a matching from L to R. You may assume Hall's Theorem. Hint: Consider the sum of degrees of the vertices in any subset S of L, and $N(S)$.

Your proof will be graded primarily on the clarity of your argument.

Problem 4 (4 points)**.** A *5-path* communication network is shown below. From this, it's easy to see what an n-path network would be.



5-Path

(a) (2 points) Circle all the permutations, mapping inputs to outputs, that ensure that the max congestion in a n-path network is at least n:

- $\pi(i) = n - 1 - i$
- $\pi(i) = \begin{cases} \lfloor n/2 \rfloor + i & \text{if } i < n/2, \\ i - \lceil n/2 \rceil & \text{if } i \geq n/2. \end{cases}$
- $\pi(i) = \begin{cases} i + 1 & \text{if } i < n - 1, \\ 0 & \text{if } i = n - 1. \end{cases}$
- $\pi(i) = i$

(b) (2 points) Explain why the max congestion in a n-path network is at most n.

Appendix

$\lceil x \rceil$ is the smallest integer y such that $y \geq x$.

$\lfloor x \rfloor$ is the largest integer y such that $y \leq x$.

Stable Marriage

A *marriage assignment* or *perfect matching* is a bijection, w : Boys $\rightarrow$ Girls.

A *rogue couple* is a boy, B, and a girl, G, such that B prefers G to his wife, and G prefers B to her husband.

An assignment is *stable* if it has no rogue couples.

Graph coloring

The *graph coloring problem* is as follows: Given a graph G, assign colors to each node such that adjacent nodes have different colors. A color assignment with this property is called a *valid coloring* of the graph —a "coloring," for short. A graph G is ***k-colorable*** if it has a coloring that uses at most k colors. The minimum value of k for which a coloring exists is called the ***chromatic number***, $\chi(G)$, of G.

Bipartite graphs

A *bipartite graph* is a graph together with a partition of its vertices into two sets, L and R, such that every edge is incident to a vertex in L and to a vertex in R.

A *bipartite matching* from L to R is a way of assigning each vertex in L to some vertex in R with which it shares an edge, and such that each vertex in R gets assigned to at most one vertex in L.

In any graph, the set $N(S)$, of *neighbors* of some set, S, of vertices is the set of all vertices adjacent to any vertex in S.

S is called a *bottleneck* if

$$|S| > |N(S)|.$$

Theorem 4.1 (Hall's Theorem)**.** *Let G be a bipartite graph with vertex partition L, R. There is total matching from L to R iff no subset of L is a bottleneck.*

A bipartite graph G with vertex partition L, R is *degree-constrained* if $\deg(l) \geq \deg(r)$ for every $l \in L$ and $r \in R$.

Communication networks

The ***congestion*** of a set of paths $P_{0,\pi(0)}, \ldots, P_{N-1,\pi(N-1)}$ is equal to the largest number of paths that pass through a single switch.

The ***max congestion*** of a network is the *maximum* over all permutations π of the *minimum* over all paths $P_{i,\pi(i)}$ of the congestion of the paths.