Mini-Quiz Feb. 28

Your name:____

Circle the name of your TA/LA:

Chiyoun Jay Jeffrey Jessica Tina

- This quiz is **closed book**. Total time is 25 minutes.
- There are five (5) problems totalling 15 points.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	3		
2	2		
3	4		
4	2		
5	4		
Total	15		

Problem 1 (3 points). For each of the following relations indicated below —the first two are on the set $\{1, 2, 3, 4\}$ —indicate whether it is **R**eflexive, **A**ntisymmetric, **TRAN**sitive, **TOT**al, or **N**one of the above. (More than one property may hold for some relations.)

$$\{(1,1),(1,3),(3,1)\}\_\_\_\_\_\_\_\_$$
$$\{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}\_\_\_\_\_\_\_\_$$
$$\{(p,q) \mid p \text{ and } q \text{ are born in the same town}\}\_\_\_\_\_\_\_\_$$

Note that every person is born in the same town as themselves.

Problem 2 (2 points). We use the notation $m \mid n$ to indicate that m divides n, and, for purposes of defining minimal elements in the divides partial order, when $m \mid n$, we'll consider m to be "smaller than or equal to" n.

Let $S ::= \{2, 6, 9, 12, 18, 27, 48, 72\}$ be partially ordered by the divides relation.

(a) **(0.5 points)** Which are the minimal elements of S?________

(b) **(0.5 points)** Which are the maximal elements?________

(c) **(0.5 points)** Give an example of a maximum-size chain in S.____________

(d) **(0.5 points)** Give an example of a maximum-size antichain in S.____________

Problem 3 (4 points). (a) (1 point) Express the sum of the first n odd numbers, $1 + 3 + 5 + \ldots$, without the dots by filling in the two missing parts in the following Σ-expression:

$$\underbrace{1 + 3 + 5 + \ldots}_{n \text{ terms}} = \Sigma_{i=0}^{(\quad)} (\qquad)$$

(b) (3 points) Prove by induction that this sum is n^2.

Problem 4 (2 points). The faulty horses-of-the-same color proof from Notes 3 appears in the box below. Among the following statements, circle the one that *best* explains the mistake in the proof. Note that most of the statements are correct, but one of them is clearly the best explanation.

1. The inductive step assumes that for any two sets of n out of $n+1$ horses, there is a horse common to both sets, but this is false for $n = 1$.

2. The proof of the inductive step is false when $n = 2$.

3. It's silly to say that a horse is the same color as itself, so the claim that the base case is "certainly" true is wrong.

4. The base case should be for $n = 0$.

5. The inductive step assumes that "being the same color" is a transitive relation on a set of 2 horses.

False Theorem 4.1. *In every set of $n \geq 1$ horses, all are the same color.*

Proof. The proof is by induction on n. The induction hypothesis, $P(n)$, will be

$$\text{In every set of } n \text{ horses, all are the same color.} \tag{1}$$

Base case: ($n = 1$). $P(1)$ is true, because in a set of horses of size 1, there's only one horse, and this horse is definitely the same color as itself.

Inductive step: Assume that $P(n)$ is true for some $n \geq 1$. that is, assume that in every set of n horses, all are the same color. Now consider a set of $n+1$ horses:

$$h_1,\ h_2,\ \ldots,\ h_n,\ h_{n+1}$$

By our assumption, the first n horses are the same color:

$$\underbrace{h_1,\ h_2,\ \ldots,\ h_n,}_{\text{same color}}\ h_{n+1}$$

Also by our assumption, the last n horses are the same color:

$$h_1,\ \underbrace{h_2,\ \ldots,\ h_n,\ h_{n+1}}_{\text{same color}}$$

So h_1 is the same color as the remaining horses besides h_{n+1}, and likewise h_{n+1} is the same color as the remaining horses besides h_1. So h_1 and h_{n+1} are the same color. That is, horses $h_1, h_2, \ldots, h_{n+1}$ must all be the same color, and so $P(n+1)$ is true. Thus, $P(n)$ implies $P(n+1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 1$. $\square$

Problem 5 (4 points). The next page contains a proof using the Well Ordering Principle that every amount of postage that can be paid exactly using only 6 cent and 15 cent stamps, is divisible by 3. That is, letting $S(n)$ mean that exactly n cents postage can be paid using only 6 and 15 cent stamps, the proof shows that

$$\forall n \in \mathbb{N}.\ S(n) \text{ implies } 3 \mid n. \qquad (*)$$

Fill in the missing portions (indicated by "…") of the following proof of (*).

So let C be the set of *counterexamples* to (*), namely

$$C ::= \{n \mid \textbf{... circle the correct choice below ...}\}$$

- $\neg[S(n)$ equiv $3 \mid n]$
- $S(n)$ and $\neg(3 \mid n)$
- $\neg S(n)$ and $\neg(3 \mid n)$
- $\neg S(n)$ or $3 \mid n$

If C is nonempty, then by the WOP, there is a smallest number, $m \in C$. This m must be positive because...

But if $S(m)$ holds and m is positive, then $S(m-6)$ or $S(m-15)$ must hold, because...

So suppose $S(m-6)$ holds. Then $3 \mid (m-6)$, because...

But if $3 \mid (m-6)$, then obviously $3 \mid m$, contradicting the fact that m is a counterexample.

Next suppose $S(m-15)$ holds. Then the proof for $m-6$ carries over directly for $m-15$ to yield a contradiction in this case as well. Since we get a contradiction in both cases, we conclude that...

which proves that (*) holds.

Appendix

Relational Properties

A binary relation, R, on a set, A, is

- total (as a relation; *not* the same as total order) if every element of the domain, A is related to some element of A:
$$\forall a_1 \in A \; \exists a_2 \in A. \; a_1 R a_2.$$
- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$,
- *reflexive* if aRa for every $a \in A$,
- *antisymmetric* if for every $a \neq b \in A$, aRb implies $\neg(bRa)$,

Partial Order

A binary relation is a *strict partial order* iff it is transitive and asymmetric. It is a *weak partial order* iff it is transitive, reflexive, and antisymmetric.

Let $\preceq$ be a weak (reflexive) partial order on a set, A.

- An element $a \in A$ is *minimal* iff there is no element in A that is $\preceq a$ except possibly a itself. Similarly, an element $a \in A$ is *maximal* iff there is no element in A that is $\succeq a$ except possibly a itself.
- An element $a \in A$ is a *lower bound* for a subset, $S \subseteq A$ iff $a \preceq s$ for every $s \in S$. Similarly, an element $a \in A$ is an *upper bound* for a subset, $S \subseteq A$ iff $s \preceq a$ for every $s \in S$.
- An element $a \in A$ is the *minimum* element iff a is a lower bound on A. Similarly, an element $a \in A$ is the *maximum* element iff a is an upper bound on A.
- Elements $a, b \in A$ are *comparable* iff either $a \preceq b$ or $b \preceq a$. Two elements are *incomparable* iff they are not comparable.
- A subset, $S \subseteq A$ is *totally ordered* iff every two distinct elements in S are comparable.
- A *chain* is a totally ordered subset of A.
- An *antichain* is a subset of A, such that no two elements in it are comparable.