More Counting

1 Counting Subsets

How many k-element subsets of an n-element set are there? This question arises all the time in various guises:

- In how many ways can I select 5 books from my collection of 100 to bring on vacation?
- How many different 13-card Bridge hands can be dealt from a 52-card deck?
- In how many ways can I select 5 toppings for my pizza if there are 14 available toppings?

This number comes up so often that there is a special notation for it:

$$\binom{n}{k} ::= \text{ the number of } k\text{-element subsets of an } n\text{-element set.}$$

The expression $\binom{n}{k}$ is read "n choose k." Now we can immediately express the answers to all three questions above:

- I can select 5 books from 100 in $\binom{100}{5}$ ways.
- There are $\binom{52}{13}$ different Bridge hands.
- There are $\binom{14}{5}$ different 5-topping pizzas, if 14 toppings are available.

1.1 The Subset Rule

We can derive a simple formula for the n-choose-k number using the Division Rule. We do this by mapping any permutation of an n-element set $\{a_1, \ldots, a_n\}$ into a k-element subset simply by taking the first k elements of the permutation. That is, the permutation $a_1 a_2 \ldots a_n$ will map to the set $\{a_1, a_2, \ldots, a_k\}$.

Notice that any other permutation with the same first k elements $a_1, \ldots, a_k$ *in any order* and the same remaining elements $n-k$ elements *in any order* will also map to this set. What's more, a permutation can only map to $\{a_1, a_2, \ldots, a_k\}$ if its first k elements are the elements $a_1, \ldots, a_k$ in some order. Since there are $k!$ possible permutations of the first k elements and $(n-k)!$ permutations of the remaining elements, we conclude from the Product Rule that exactly $k!(n-k)!$ permutations of the n-element set map to the the particular subset, S. In other words, the mapping from permutations to k-element subsets is $k!(n-k)!$-to-1.

But we know there are $n!$ permutations of an n-element set, so by the Division Rule, we conclude that

$$n! = k!(n-k)!\binom{n}{k}$$

which proves:

Rule 1 (Subset Rule)**.** *The number,*

$$\binom{n}{k},$$

of k-element subsets of an n-element set is

$$\frac{n!}{k!\,(n-k)!}.$$

Notice that this works even for 0-element subsets: $n!/0!n! = 1$. Here we use the fact that $0!$ is a *product* of 0 terms, which by convention equals 1. (A *sum* of zero terms equals 0.)

1.2 Bit Sequences

How many n-bit sequences contain exactly k ones? We've already seen the straightforward bijection between subsets of an n-element set and n-bit sequences. For example, here is a 3-element subset of $\{x_1, x_2, \ldots, x_8\}$ and the associated 8-bit sequence:

$$\begin{array}{cccccccccc} \{ & x_1, & & & x_4, & x_5 & & & & \} \\ (& 1, & 0, & 0, & 1, & 1, & 0, & 0, & 0 &) \end{array}$$

Notice that this sequence has exactly 3 ones, each corresponding to an element of the 3-element subset. More generally, the n-bit sequences corresponding to a k-element subset will have exactly k ones. So by the Bijection Rule,

> The number of n-bit sequences with exactly k ones is $\binom{n}{k}$.

2 Magic Trick

There is a Magician and an Assistant. The Assistant goes into the audience with a deck of 52 cards while the Magician looks away. Five audience members each select one card from the deck. The Assistant then gathers up the five cards and holds up four of them so the Magician can see them. The Magician concentrates for a short time and then correctly names the secret, fifth card!

Since we don't really believe the Magician can read minds, we know the Assistant has somehow communicated the secret card to the Magician. Since real Magicians and Assistants are not to be trusted, we can expect that the Assistant would illegitimately signal the Magician with coded phrases or body language, but they don't have to cheat in this way. In fact, the Magician and Assistant could be kept out of sight of each other while some audience member holds up the 4 cards designated by the Assistant for the Magician to see.

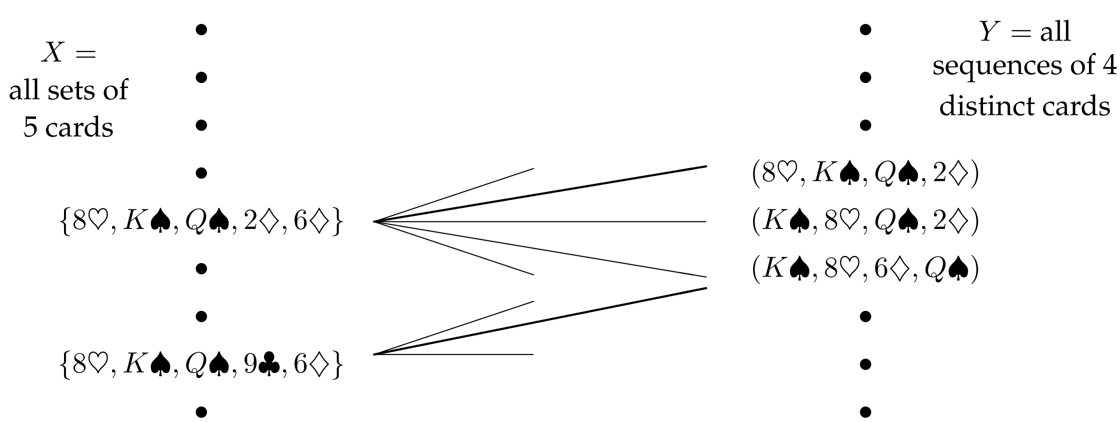
Of course, without cheating, there is still an obvious way the Assistant can communicate to the Magician: he can choose any of the $4! = 24$ permutations of the 4 cards as the order in which to hold up the cards. However, this alone won't quite work: there are 48 cards remaining in the deck, so the Assistant doesn't have enough choices of orders to indicate exactly what the secret card is (though he could narrow it down to two cards).

2.1 The Secret

The method the Assistant can use to communicate the fifth card exactly is a nice application of what we know about counting and matching.

The Assistant really has another legitimate ways to communicate: he can choose *which of the five cards to keep hidden*. Of course, it's not clear how the Magician could determine which of these five possibilities the Assistant selected by looking at the four visible cards, but there is a way as we'll now explain.

The problem facing the Magician and Assistant is actually a bipartite matching problem. Put all the *sets* of 5 cards in a collection X on the left. And put all the sequences of 4 distinct cards in a collection Y on the right. These are the two sets of vertices in the bipartite graph. There is an edge between a set of 5 cards and a sequence of 4 if every card in the sequence is also in the set. In other words, if the audience selects a set of cards, then the Assistant must reveal a sequence of cards that is adjacent in the bipartite graph. Some edges are shown in the diagram below.



For example, $\{8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit, 6\diamondsuit\}$ is an element of X on the left. If the audience selects this set of 5 cards, then there are many different 4-card sequences on the right in set Y that the Assistant could choose to reveal, including $(8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit)$, $(K\spadesuit, 8\heartsuit, Q\spadesuit, 2\diamondsuit)$, and $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$.

What the Magician and his Assistant need to perform the trick is a *matching* for the X vertices. If they agree in advance on some matching, then when the audience selects a set of 5 cards, the

Assistant reveals the matching sequence of 4 cards. The Magician uses the reverse of the matching to find the audience's chosen set of 5 cards, and so he can name the one not already revealed.

For example, suppose the Assistant and Magician agree on a matching containing the two bold edges in the diagram above. If the audience selects the set $\{8\heartsuit, K\spadesuit, Q\spadesuit, 9\clubsuit, 6\diamondsuit\}$, then the Assistant reveals the corresponding sequence $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$. Using the matching, the Magician sees that $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$ is indeed matched to $\{8\heartsuit, K\spadesuit, Q\spadesuit, 9\clubsuit, 6\diamondsuit\}$, so he can name the one card in the corresponding set not already revealed, namely, the $9\clubsuit$. Notice that the fact that the sets are *matched*, that is, that different sets are paired with *distinct* sequences, is essential. For example, the Assistant could have revealed the same sequence, $(K\spadesuit, 8\heartsuit, 6\diamondsuit, Q\spadesuit)$, if the audience picked a different set $\{8\heartsuit, K\spadesuit, Q\spadesuit, 2\diamondsuit, 6\diamondsuit\}$, but if that happened, then the Magician would have no way to tell if remaining card was the $9\clubsuit$ or $2\diamondsuit$.

So how can we be sure the needed matching can be found? The reason is that each vertex on the left has degree $5 \cdot 4! = 120$, since there are five ways to select the card kept secret and there are $4!$ permutations of the remaining 4 cards. In addition, each vertex on the right has degree 48, since there are 48 possibilities for the fifth card. So this graph is *degree-constrained* (see Notes 6), and therefore satisfies Hall's matching condition.

In fact, this reasoning show that the Magician could still pull off the trick if 120 cards were left instead of 48, that is, the trick would work with a deck as large as 124 different cards —without any magic!

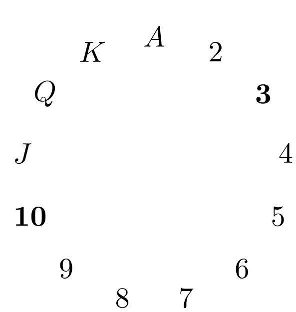
2.2 The Real Secret

But wait a minute! It's all very well in principle to have the Magician and his Assistant agree on a matching, but how are they supposed to remember a matching with $\binom{52}{5} = 2,598,960$ edges? For the trick to work in practice, there has to be a way to match hands and card sequences mentally and on the fly.

We'll describe one approach. As a running example, suppose that the audience selects:

$$10\heartsuit \quad 9\diamondsuit \quad 3\heartsuit \quad Q\spadesuit \quad J\diamondsuit$$

- The Assistant picks out two cards of the same suit. In the example, the assistant might choose the $3\heartsuit$ and $10\heartsuit$.
- The Assistant locates the values of these two cards on the cycle shown below:



For any two distinct values on this cycle, one is always between 1 and 6 hops clockwise from the other. For example, the $3\heartsuit$ is 6 hops clockwise from the $10\heartsuit$.

- The more counterclockwise of these two cards is revealed first, and the other becomes the secret card. Thus, in our example, the $10\heartsuit$ would be revealed, and the $3\heartsuit$ would be the secret card. Therefore:
  - The suit of the secret card is the same as the suit of the first card revealed.
  - The value of the secret card is between 1 and 6 hops clockwise from the value of the first card revealed.
- All that remains is to communicate a number between 1 and 6. The Magician and Assistant agree beforehand on an ordering of all the cards in the deck from smallest to largest such as:

  $$A\clubsuit\ 2\clubsuit\ \ldots\ K\clubsuit\ A\diamondsuit\ 2\diamondsuit\ \ldots\ K\diamondsuit\ A\heartsuit\ 2\heartsuit\ \ldots\ K\heartsuit\ A\spadesuit\ 2\spadesuit\ \ldots\ K\spadesuit$$

  The order in which the last three cards are revealed communicates the number according to the following scheme:

  $$\begin{array}{ccccccc} (& \text{small}, & \text{medium}, & \text{large} &) & = 1 \\ (& \text{small}, & \text{large}, & \text{medium} &) & = 2 \\ (& \text{medium}, & \text{small}, & \text{large} &) & = 3 \\ (& \text{medium}, & \text{large}, & \text{small} &) & = 4 \\ (& \text{large}, & \text{small}, & \text{medium} &) & = 5 \\ (& \text{large}, & \text{medium}, & \text{small} &) & = 6 \end{array}$$

  In the example, the Assistant wants to send 6 and so reveals the remaining three cards in large, medium, small order. Here is the complete sequence that the Magician sees:

  $$10\heartsuit \quad Q\spadesuit \quad J\diamondsuit \quad 9\diamondsuit$$

- The Magician starts with the first card, $10\heartsuit$, and hops 6 values clockwise to reach $3\heartsuit$, which is the secret card!

So that's how the trick can work with a standard deck of 52 cards. On the other hand, Hall's Theorem implied that the Magician and Assistant could in principle perform the trick with a deck of up to 124 cards. Until very recently we didn't know how to perform it in practice for decks larger than the standard one, but on March 30, 2007, David Shin, who was 6.042 TA in Spring '06, sent an email describing an ingenious matching for a 124 card deck that a Magician and Assistant could actually figure out in their heads. We'll describe Shin's method another time.

2.3 Same Trick with Four Cards?

Suppose that the audience selects only *four* cards and the Assistant reveals a sequence of *three* to the Magician. Can the Magician determine the fourth card?

Let X be all the sets of four cards that the audience might select, and let Y be all the sequences of three cards that the Assistant might reveal. Now, on one hand, we have

$$|X| = \binom{52}{4} = 270,725$$

by the Subset Rule. On the other hand, we have

$$|Y| = 52 \cdot 51 \cdot 50 = 132,600$$

by the Generalized Product Rule. Thus, by the Pigeonhole Principle, the Assistant must reveal the *same* sequence of three cards for some two *different* sets of four. This is bad news for the Magician: if he sees that sequence of three, then there are at least two possibilities for the fourth card which he cannot distinguish. So there is no legitimate way for the Assistant to communicate exactly what the fourth card is!

3 The Bookkeeper Rule

3.1 Sequences of Subsets

Choosing a k-element subset of an n-element set is the same as splitting the set into a pair of subsets: the first subset of size k and the second subset consisting of the remaining $n-k$ elements. So the Subset Rule can be understood as a rule for counting the number of such splits into pairs of subsets.

We can generalize this to splits into m subsets. Namely, let A be an n-element set and $k_1, k_2, \ldots, k_m$ be nonnegative integers whose sum is n. A $(k_1, k_2, \ldots, k_m)$***-split of*** A is a sequence

$$(A_1, A_2, \ldots, A_m)$$

where the A_i are pairwise disjoint subsets of A and $|A_i| = k_i$ for $i = 1, \ldots, m$.

The same reasoning used to explain the Subset Rule extends directly to a rule for counting the number of splits into subsets of given sizes.

Rule 2 (Subset Split Rule)**.** *The number of* $(k_1, k_2, \ldots, k_m)$*-splits of an* n*-element set is*

$$\binom{n}{k_1, \ldots, k_m} ::= \frac{n!}{k_1!\, k_2!\, \cdots k_m!}$$

The proof of this Rule is essentially the same as for the Subset Rule. Namely, we map any permutation $a_1 a_2 \ldots a_n$ of an n-element set, A, into a $(k_1, k_2, \ldots, k_m)$-split by letting the 1st subset in the split be the first k_1 elements of the permutation, the 2nd subset of the split be the next k_2 elements, ..., and the mth subset of the split be the final k_m elements of the permutation. This map is a $k_1!\, k_2!\, \cdots\, k_m!$-to-1 from the $n!$ permutations to the $(k_1, k_2, \ldots, k_m)$-splits of A, and the Subset Split Rule now follows from the Division Rule.

3.2 Sequences over an alphabet

We can also generalize our count of n-bit sequences with k-ones to counting length n sequences of letters over an alphabet with more than two letters. For example, how many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

Notice that there are 1 B, 2 O's, 2 K's, 3 E's, 1 P, and 1 R in BOOKKEEPER. This leads to a straightforward bijection between permutations of BOOKKEEPER and (1,2,2,3,1,1)-splits of $\{1, \ldots, n\}$.

Namely, map a permutation to the sequence of sets of positions where each of the different letters occur.

For example, in the permutation BOOKKEEPER itself, the B is in the 1st position, the O's occur in the 2nd and 3rd positions, K's in 4th and 5th, the E's in the 6th, 7th and 9th, P in the 8th, and R is in the 10th position, so BOOKKEEPER maps to

$$(\{1\}, \{2,3\}, \{4,5\}, \{6,7,9\}, \{8\}, \{10\}).$$

From this bijection and the Subset Split Rule, we conclude that the number of ways to rearrange the letters in the word BOOKKEEPER is:

$$\frac{\overbrace{10!}^{\text{total letters}}}{\underbrace{1!}_{\text{B's}}\underbrace{2!}_{\text{O's}}\underbrace{2!}_{\text{K's}}\underbrace{3!}_{\text{E's}}\underbrace{1!}_{\text{P's}}\underbrace{1!}_{\text{R's}}}$$

This example generalizes directly to an exceptionally useful counting principle which we will call the

Rule 3 (Bookkeeper Rule)**.** *Let $l_1, \ldots, l_m$ be distinct elements. The number of sequences with k_1 occurrences of l_1, and k_2 occurrences of l_2, ..., and k_m occurrences of l_m is*

$$\frac{(k_1 + k_2 + \ldots + k_m)!}{k_1!\, k_2!\, \ldots\, k_m!}$$

Example. 20-Mile Walks.

I'm planning a 20-mile walk, which should include 5 northward miles, 5 eastward miles, 5 southward miles, and 5 westward miles. How many different walks are possible?

There is a bijection between such walks and sequences with 5 N's, 5 E's, 5 S's, and 5 W's. By the Bookkeeper Rule, the number of such sequences is:

$$\frac{20!}{5!^4}$$

3.3 A Word about Words

Someday you might refer to the Subset Split Rule or the Bookkeeper Rule in front of a roomful of colleagues and discover that they're all staring back at you blankly. This is not because they're dumb, but rather because we made up the name "Bookkeeper Rule". However, the rule is excellent and the name is apt, so we suggest that you play through: "You know? The Bookkeeper Rule? Don't you guys know *anything???*"

The Bookkeeper Rule is sometimes called the "formula for permutations with indistinguishable objects." The size k subsets of an n-element set are sometimes called ***k-combinations***. Other similar-sounding descriptions are "combinations with repetition, permutations with repetition, r-permutations, permutations with indistinguishable objects," and so on. However, the counting rules we've taught you are sufficient to solve all these sorts of problems without knowing this jargon, so we'll skip it.

4 Poker Hands

There are 52 cards in a deck. Each card has a *suit* and a *value*. There are four suits:

$$\overset{\text{spades}}{\spadesuit} \qquad \overset{\text{hearts}}{\heartsuit} \qquad \overset{\text{clubs}}{\clubsuit} \qquad \overset{\text{diamonds}}{\diamondsuit}$$

And there are 13 values:

$$2\,,\ 3\,,\ 4\,,\ 5\,,\ 6\,,\ 7\,,\ 8\,,\ 9\,,\ \overset{\text{jack}}{J}\,,\ \overset{\text{queen}}{Q}\,,\ \overset{\text{king}}{K}\,,\ \overset{\text{ace}}{A}$$

Thus, for example, $8\heartsuit$ is the 8 of hearts and $A\spadesuit$ is the ace of spades. Values farther to the right in this list are considered "higher" and values to the left are "lower".

Five-Card Draw is a card game in which each player is initially dealt a *hand*, a subset of 5 cards. (Then the game gets complicated, but let's not worry about that.) The number of different hands in Five-Card Draw is the number of 5-element subsets of a 52-element set, which is 52 choose 5:

$$\text{total \# of hands} = \binom{52}{5} = 2,598,960$$

Let's get some counting practice by working out the number of hands with various special properties.

4.1 Hands with a Four-of-a-Kind

A *Four-of-a-Kind* is a set of four cards with the same value. How many different hands contain a Four-of-a-Kind? Here are a couple examples:

$$\begin{array}{cccccc} \{ & 8\spadesuit, & 8\diamondsuit, & Q\heartsuit, & 8\heartsuit, & 8\clubsuit & \} \\ \{ & A\clubsuit, & 2\clubsuit, & 2\heartsuit, & 2\diamondsuit, & 2\spadesuit & \} \end{array}$$

As usual, the first step is to map this question to a sequence-counting problem. A hand with a Four-of-a-Kind is completely described by a sequence specifying:

1. The value of the four cards.
2. The value of the extra card.
3. The suit of the extra card.

Thus, there is a bijection between hands with a Four-of-a-Kind and sequences consisting of two distinct values followed by a suit. For example, the three hands above are associated with the following sequences:

$$\begin{array}{ccccccccc} (8, Q, \heartsuit) & \leftrightarrow & \{ & 8\spadesuit, & 8\diamondsuit, & 8\heartsuit, & 8\clubsuit, & Q\heartsuit & \} \\ (2, A, \clubsuit) & \leftrightarrow & \{ & 2\clubsuit, & 2\heartsuit, & 2\diamondsuit, & 2\spadesuit, & A\clubsuit & \} \end{array}$$

Now we need only count the sequences. There are 13 ways to choose the first value, 12 ways to choose the second value, and 4 ways to choose the suit. Thus, by the Generalized Product Rule, there are $13 \cdot 12 \cdot 4 = 624$ hands with a Four-of-a-Kind. This means that only 1 hand in about 4165 has a Four-of-a-Kind; not surprisingly, this is considered a very good poker hand!

4.2 Hands with a Full House

A *Full House* is a hand with three cards of one value and two cards of another value. Here are some examples:

$$\begin{array}{c} \{\ 2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\ \} \\ \{\ 5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\ \} \end{array}$$

Again, we shift to a problem about sequences. There is a bijection between Full Houses and sequences specifying:

1. The value of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways.
3. The value of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways.

The example hands correspond to sequences as shown below:

$$\begin{array}{ccc} (2, \{\spadesuit, \clubsuit, \diamondsuit\}, J, \{\clubsuit, \diamondsuit\}) & \leftrightarrow & \{\ 2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\ \} \\ (5, \{\diamondsuit, \clubsuit, \heartsuit\}, 7, \{\heartsuit, \clubsuit\}) & \leftrightarrow & \{\ 5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\ \} \end{array}$$

By the Generalized Product Rule, the number of Full Houses is:

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

We're on a roll— but we're about to hit a speedbump.

4.3 Hands with Two Pairs

How many hands have *Two Pairs*; that is, two cards of one value, two cards of another value, and one card of a third value? Here are examples:

$$\begin{array}{c} \{\ 3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\ \} \\ \{\ 9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\ \} \end{array}$$

Each hand with Two Pairs is described by a sequence consisting of:

1. The value of the first pair, which can be chosen in 13 ways.
2. The suits of the first pair, which can be selected $\binom{4}{2}$ ways.
3. The value of the second pair, which can be chosen in 12 ways.
4. The suits of the second pair, which can be selected in $\binom{4}{2}$ ways.
5. The value of the extra card, which can be chosen in 11 ways.
6. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

Thus, it might appear that the number of hands with Two Pairs is:

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4$$

Wrong answer! The problem is that there is *not* a bijection from such sequences to hands with Two Pairs. This is actually a 2-to-1 mapping. For example, here are the pairs of sequences that map to the hands given above:

$$\begin{array}{ll} (3, \{\diamondsuit, \spadesuit\}, Q, \{\diamondsuit, \heartsuit\}, A, \clubsuit) & \searrow \\ & \quad \{ \quad 3\diamondsuit, \quad 3\spadesuit, \quad Q\diamondsuit, \quad Q\heartsuit, \quad A\clubsuit \quad \} \\ (Q, \{\diamondsuit, \heartsuit\}, 3, \{\diamondsuit, \spadesuit\}, A, \clubsuit) & \nearrow \\ (9, \{\heartsuit, \diamondsuit\}, 5, \{\heartsuit, \clubsuit\}, K, \spadesuit) & \searrow \\ & \quad \{ \quad 9\heartsuit, \quad 9\diamondsuit, \quad 5\heartsuit, \quad 5\clubsuit, \quad K\spadesuit \quad \} \\ (5, \{\heartsuit, \clubsuit\}, 9, \{\heartsuit, \diamondsuit\}, K, \spadesuit) & \nearrow \end{array}$$

The problem is that nothing distinguishes the first pair from the second. A pair of 5's and a pair of 9's is the same as a pair of 9's and a pair of 5's. We avoided this difficulty in counting Full Houses because, for example, a pair of 6's and a triple of kings is different from a pair of kings and a triple of 6's.

We ran into precisely this difficulty last time, when we went from counting arrangements of *different* pieces on a chessboard to counting arrangements of two *identical* rooks. The solution then was to apply the Division Rule, and we can do the same here. In this case, the Division rule says there are twice as many sequences as hands, so the number of hands with Two Pairs is actually:

$$\frac{13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4}{2}$$

Another Approach

The preceding example was disturbing! One could easily overlook the fact that the mapping was 2-to-1 on an exam, fail the course, and turn to a life of crime. You can make the world a safer place in two ways:

1. Whenever you use a mapping $f : A \to B$ to translate one counting problem to another, check the number of elements in A that are mapped to each element in B. This determines the size of A relative to B. You can then apply the Division Rule with the appropriate correction factor.

2. As an extra check, try solving the same problem in a different way. Multiple approaches are often available— and all had better give the same answer! (Sometimes different approaches give answers that *look* different, but turn out to be the same after some algebra.)

We already used the first method; let's try the second. There is a bijection between hands with two pairs and sequences that specify:

1. The values of the two pairs, which can be chosen in $\binom{13}{2}$ ways.

2. The suits of the lower-value pair, which can be selected in $\binom{4}{2}$ ways.
3. The suits of the higher-value pair, which can be selected in $\binom{4}{2}$ ways.
4. The value of the extra card, which can be chosen in 11 ways.
5. The suit of the extra card, which can be selected in $\binom{4}{1} = 4$ ways.

For example, the following sequences and hands correspond:

$$\begin{array}{ccl} (\{3, Q\}, \{\diamondsuit, \spadesuit\}, \{\diamondsuit, \heartsuit\}, A, \clubsuit) & \leftrightarrow & \{\ 3\diamondsuit,\ 3\spadesuit,\ Q\diamondsuit,\ Q\heartsuit,\ A\clubsuit\ \} \\ (\{9, 5\}, \{\heartsuit, \clubsuit\}, \{\heartsuit, \diamondsuit\}, K, \spadesuit) & \leftrightarrow & \{\ 9\heartsuit,\ 9\diamondsuit,\ 5\heartsuit,\ 5\clubsuit,\ K\spadesuit\ \} \end{array}$$

Thus, the number of hands with two pairs is:

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4$$

This is the same answer we got before, though in a slightly different form.

4.4 Hands with Every Suit

How many hands contain at least one card from every suit? Here is an example of such a hand:

$$\{\ 7\diamondsuit,\ K\clubsuit,\ 3\diamondsuit,\ A\heartsuit,\ 2\spadesuit\ \}$$

Each such hand is described by a sequence that specifies:

1. The values of the diamond, the club, the heart, and the spade, which can be selected in $13 \cdot 13 \cdot 13 \cdot 13 = 13^4$ ways.
2. The suit of the extra card, which can be selected in 4 ways.
3. The value of the extra card, which can be selected in 12 ways.

For example, the hand above is described by the sequence:

$$(7, K, A, 2, \diamondsuit, 3) \quad \leftrightarrow \quad \{\ 7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\ \}$$

Are there other sequences that correspond to the same hand? There is one more! We could equally well regard either the $3\diamondsuit$ or the $7\diamondsuit$ as the extra card, so this is actually a 2-to-1 mapping. Here are the two sequences corresponding to the example hand:

$$\begin{array}{cc} (7, K, A, 2, \diamondsuit, 3) \ \searrow & \\ & \{\ 7\diamondsuit,\ K\clubsuit,\ A\heartsuit,\ 2\spadesuit,\ 3\diamondsuit\ \} \\ (3, K, A, 2, \diamondsuit, 7) \ \nearrow & \end{array}$$

Therefore, the number of hands with every suit is:

$$\frac{13^4 \cdot 4 \cdot 12}{2}$$

5 Binomial Theorem

Counting gives insight into one of the basic theorems of algebra. A ***binomial*** is a sum of two terms, such as $a + b$. Now consider its 4th power, $(a + b)^4$.

If we multiply out this 4th power expression completely, we get

$$\begin{array}{rcccccccc} (a+b)^4 & = & aaaa & + & aaab & + & aaba & + & aabb \\ & + & abaa & + & abab & + & abba & + & abbb \\ & + & baaa & + & baab & + & baba & + & babb \\ & + & bbaa & + & bbab & + & bbba & + & bbbb \end{array}$$

Notice that there is one term for every sequence of a's and b's. So there are 2^4 terms, and the number of terms with k copies of b and $n - k$ copies of a is:

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

by the Bookkeeper Rule. Now let's group equivalent terms, such as $aaab = aaba = abaa = baaa$. Then the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. So for $n = 4$, this means:

$$(a+b)^4 = \binom{4}{0} \cdot a^4 b^0 + \binom{4}{1} \cdot a^3 b^1 + \binom{4}{2} \cdot a^2 b^2 + \binom{4}{3} \cdot a^1 b^3 + \binom{4}{4} \cdot a^0 b^4$$

In general, this reasoning gives the Binomial Theorem:

Theorem 5.1 (Binomial Theorem)**.** *For all $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

The expression $\binom{n}{k}$ is often called a "binomial coefficient" in honor of its appearance here.

This reasoning about binomials extends nicely to ***multinomials***, which are sums of two or more terms. For example, suppose we wanted the coefficient of

$$bo^2k^2e^3pr$$

in the expansion of $(b+o+k+e+p+r)^{10}$. Each term in this expansion is a product of 10 variables where each variable is one of b, o, k, e, p, or r. Now, the coefficient of $bo^2k^2e^3pr$ is the number of those terms with exactly 1 b, 2 o's, 2 k's, 3 e's, 1 p, and 1 r. And the number of such terms is precisely the number of rearrangments of the word BOOKKEEPER:

$$\binom{10}{1,2,2,3,1,1} = \frac{10!}{1!\,2!\,2!\,3!\,1!\,1!}.$$

The expression on the left is called a "multinomial coefficient." This reasoning extends to a general theorem:

Theorem 5.2 (Multinomial Theorem)**.** *For all $n \in \mathbb{N}$ and $z_1, \ldots z_m \in \mathbb{R}$:*

$$(z_1 + z_2 + \cdots + z_m)^n = \sum_{\substack{k_1,\ldots,k_m \in \mathbb{N} \\ k_1+\cdots+k_m = n}} \binom{n}{k_1, k_2, \ldots, k_m} z_1^{k_1} z_2^{k_2} \cdots z_m^{k_m}$$

You'll be better off remembering the reasoning behind the Multinomial Theorem rather than this ugly formal statement.

6 Combinatorial Proof

Suppose you have n different T-shirts, but only want to keep k. You could equally well select the k shirts you want to keep or select the complementary set of $n - k$ shirts you want to throw out. Thus, the number of ways to select k shirts from among n must be equal to the number of ways to select $n - k$ shirts from among n. Therefore:

$$\binom{n}{k} = \binom{n}{n-k}$$

This is easy to prove algebraically, since both sides are equal to:

$$\frac{n!}{k!\,(n-k)!}$$

But we didn't really have to resort to algebra; we just used counting principles.

Hmm.

6.1 Boxing

Jay, famed 6.042 TA, has decided to try out for the US Olympic boxing team. After all, he's watched all of the *Rocky* movies and spent hours in front of a mirror sneering, "Yo, you wanna piece a' *me*?!" Jay figures that n people (including himself) are competing for spots on the team and only k will be selected. As part of maneuvering for a spot on the team, he needs to work out how many different teams are possible. There are two cases to consider:

- Jay *is* selected for the team, and his $k - 1$ teammates are selected from among the other $n - 1$ competitors. The number of different teams that can be formed in this way is:

$$\binom{n-1}{k-1}$$

- Jay is *not* selected for the team, and all k team members are selected from among the other $n - 1$ competitors. The number of teams that can be formed this way is:

$$\binom{n-1}{k}$$

All teams of the first type contain Jay, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of possible Olympic boxing teams is:

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Chiyoun, equally-famed 6.042 TA, thinks Jay isn't so tough and so he might as well also try out. He reasons that n people (including himself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}$$

Chiyoun and Jay each correctly counted the number of possible boxing teams; thus, their answers must be equal. So we know:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

This is called ***Pascal's Identity***. And we proved it *without any algebra!* Instead, we relied purely on counting techniques.

6.2 Finding a Combinatorial Proof

A ***combinatorial proof*** is an argument that establishes an algebraic fact by relying on counting principles. Many such proofs follow the same basic outline:

1. Define a set S.
2. Show that $|S| = n$ by counting one way.
3. Show that $|S| = m$ by counting another way.
4. Conclude that $n = m$.

In the preceding example, S was the set of all possible Olympic boxing teams. Jay computed

$$|S| = \binom{n-1}{k-1} + \binom{n-1}{k}$$

by counting one way, and Chiyoun computed

$$|S| = \binom{n}{k}$$

by counting another. Equating these two expressions gave Pascal's Identity.

More typically, the set S is defined in terms of simple sequences or sets rather than an elaborate story. Here is less colorful example of a combinatorial argument.

Theorem 6.1.

$$\sum_{r=0}^{n} \binom{n}{r}\binom{2n}{n-r} = \binom{3n}{n}$$

Proof. We give a combinatorial proof. Let S be all n-card hands that can be dealt from a deck containing n red cards (numbered $1, \ldots, n$) and $2n$ black cards (numbered $1, \ldots, 2n$). First, note that every $3n$-element set has

$$|S| = \binom{3n}{n}$$

n-element subsets.

From another perspective, the number of hands with exactly r red cards is

$$\binom{n}{r}\binom{2n}{n-r}$$

since there are $\binom{n}{r}$ ways to choose the r red cards and $\binom{2n}{n-r}$ ways to choose the $n - r$ black cards. Since the number of red cards can be anywhere from 0 to n, the total number of n-card hands is:

$$|S| = \sum_{r=0}^{n} \binom{n}{r} \binom{2n}{n-r}$$

Equating these two expressions for $|S|$ proves the theorem. $\square$

Combinatorial proofs are almost magical. Theorem 6.1 looks pretty scary, but we proved it without any algebraic manipulations at all. The key to constructing a combinatorial proof is choosing the set S properly, which can be tricky. Generally, the simpler side of the equation should provide some guidance. For example, the right side of Theorem 6.1 is $\binom{3n}{n}$, which suggests choosing S to be all n-element subsets of some $3n$-element set.