In-Class Problems Week 9, Wed.

Problem 1. Prove that asymptotic equality ($\sim$) is a symmetric and transitive relation.

Problem 2. Recall that for functions f, g on the natural numbers, $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)| \,. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest natural number*, c, and for that smallest c, the *smallest corresponding natural number* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ ______ n_0 = ______

$g = O(f)$ YES NO If yes, $c =$ ______ n_0 = ______

Problem 3. Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants. Be prepared to justify your answers.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin n\pi/2}$						
$\log(n!)$	$\log(n^n)$						
n^k	c^n						
$\log^k n$	n^ϵ						

Problem 4. It is a standard fallacy to think that given n quantities each of which is $O(1)$, their sum would have to be $O(n)$.

Namely, let $f_1, f_2, \ldots$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$, and let

$$S(n) ::= \sum_{i=1}^{n} f_i(n).$$

Then given that $f_i = O(1)$ for every f_i in the sequence, we can try to argue as follows:

$$S(n) = \sum_{i=1}^{n} f_i(n) = \sum_{i=1}^{n} O(1) = n \cdot O(1) = O(n).$$

This informal argument may seem plausible, but is fundamentally flawed because it treats O(1) as some kind numerical quantity. In fact, we ask you to show that there is no way to determine how fast the sum, $S(n)$, may grow.

Namely, let g be any function on $\mathbb{N}$. Explain how to define a sequence of functions $f_1, f_2, \ldots$ such that each $f_i = O(1)$, but S is not $O(g)$. *Hint:* Let $f_i(n) ::= i \cdot g(i)$.

Asymptotic Notations

Lemma (Stirling's Formula)**.**

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n},$$

For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically equal* to g, in symbols,

$$f(x) \sim g(x)$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 1.$$

For functions $f, g : \mathbb{R} \to \mathbb{R}$, we say f is *asymptotically smaller* than g, in symbols,

$$f(x) = o(g(x)),$$

iff

$$\lim_{x\to\infty} f(x)/g(x) = 0.$$

Given functions $f, g : \mathbb{R} \mapsto \mathbb{R}$, with g nonnegative, we say that[1]

$$f = O(g)$$

iff

$$\limsup_{x\to\infty} |f(x)| / g(x) < \infty.$$

An alternative, equivalent, definition is

$$f = O(g)$$

iff there exists a constant $c \geq 0$ and an x_0 such that for all $x \geq x_0$, $|f(x)| \leq cg(x)$.

Finally, we say

$$f = \Theta(g) \quad \text{iff} \quad f = O(g) \wedge g = O(f).$$

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[1]

$$\limsup_{x\to\infty} h(x) ::= \lim_{x\to\infty} \text{lub}_{y \geq x} h(y).$$