

## In-Class Problems Week 8, Fri.

**Problem 1.** We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour  $1/3$  of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour  $1/3$  of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of  $n$  times.

- (a) Describe a closed form formula for the amount of wine in the first glass after  $n$  back-and-forth pourings.
- (b) What is the limit of the amount of wine in each glass as  $n$  approaches infinity?

**Problem 2.** Suppose you were about to enter college today and a college loan officer offered you the following deal: \$25,000 at the start of each year for four years to pay for your college tuition and an option of choosing one of the following repayment plans:

**Plan A:** Wait four years, then repay \$20,000 at the start of each year for the next ten years.

**Plan B:** Wait five years, then repay \$30,000 at the start of each year for the next five years.

Suppose the annual interest rate paid by banks is 7% and does not change in the future.

- (a) Assuming that it's no hardship for you to meet the terms of either payback plan, which one is a better deal? (You will need a calculator.)
- (b) What is the loan officer's effective profit (in today's dollars) on the loan?

**Problem 3.** Riemann's Zeta Function  $\zeta(k)$  is defined to be the infinite summation:

$$1 + \frac{1}{2^k} + \frac{1}{3^k} \cdots = \sum_{j \geq 1} \frac{1}{j^k}$$

Below is a proof that

$$\sum_{k \geq 2} (\zeta(k) - 1) = 1$$

Justify each line of the proof. (P.S. The purpose of this exercise is to highlight some of the rules for manipulating series. Don't worry about the significance of this identity.)

$$\sum_{k \geq 2} (\zeta(k) - 1) = \sum_{k \geq 2} \left[ \left( \sum_{j \geq 1} \frac{1}{j^k} \right) - 1 \right] \quad (1)$$

$$= \sum_{k \geq 2} \sum_{j \geq 2} \frac{1}{j^k} \quad (2)$$

$$= \sum_{j \geq 2} \sum_{k \geq 2} \frac{1}{j^k} \quad (3)$$

$$= \sum_{j \geq 2} \frac{1}{j^2} \sum_{k \geq 0} \frac{1}{j^k} \quad (4)$$

$$= \sum_{j \geq 2} \frac{1}{j^2} \cdot \frac{1}{1 - 1/j} \quad (5)$$

$$= \sum_{j \geq 2} \frac{1}{j(j-1)} \quad (6)$$

$$= \lim_{n \rightarrow \infty} \sum_{j=2}^n \frac{1}{j(j-1)} \quad (7)$$

$$= \lim_{n \rightarrow \infty} \sum_{j=2}^n \frac{1}{j-1} - \frac{1}{j} \quad (8)$$

$$= \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n} \right) \quad (9)$$

$$= 1 \quad (10)$$