

In-Class Problems Week 7, Mon.

Problem 1. Figures 1–4 show different pictures of planar graphs.

- (a) For each picture, describe its discrete faces (simple cycles that define the region borders).
- (b) Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*? – that is, they have the same discrete faces.
- (c) Describe a way to construct the embedding in Figure 4 according to the recursive definition of planar embedding (in the Appendix).

Problem 2. Prove the following assertions by structural induction on the definition of planar embedding.

- (a) In a planar embedding of a graph, each edge is traversed a total of two times by the faces of the embedding.
- (b) In a planar embedding of a graph with at least three vertices, each face is of length at least three.

Problem 3. (a) Show that if a connected planar graph with more than two vertices is bipartite, then

$$e \leq 2v - 4. \tag{1}$$

Hint: Similar to the proof that $e \leq 3v - 6$ (see the Appendix).

- (b) Conclude that that $K_{3,3}$ is not planar. ($K_{3,3}$ is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)

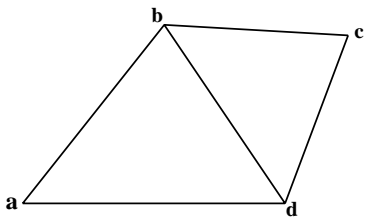


Figure 1

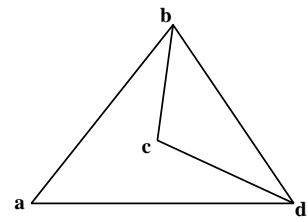


Figure 2

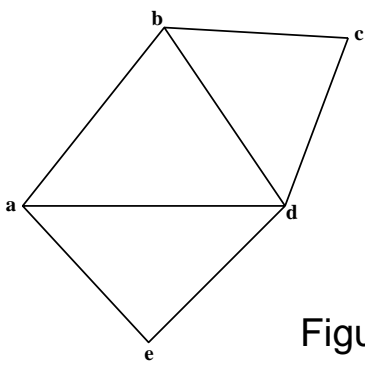


Figure 3

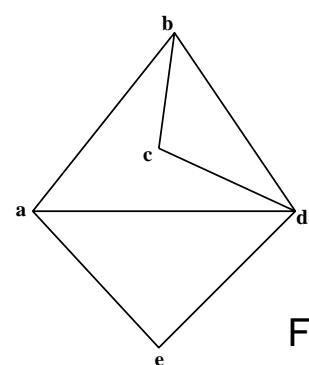


Figure 4

1 Appendix

Definition 3.1. A *planar embedding* of a *connected* graph consists of a nonempty set of cycles of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

- **Base case:** If G is a graph consisting of a single vertex, v , then a planar embedding of G has one discrete face, namely the length zero cycle, v .
- **Constructor Case:** (split a face) Suppose G is a connected graph with a planar embedding, and suppose a and b are distinct, nonadjacent vertices of G that appear on some discrete face, γ , of the planar embedding. That is, γ is a cycle of the form

$$a \dots b \dots a.$$

Then the graph obtained by adding the edge $a-b$ to the edges of G has a planar embedding with the same discrete faces as G , except that face γ is replaced by the two discrete faces¹

$$a \dots ba \quad \text{and} \quad ab \dots a,$$

as illustrated in Figure 1.

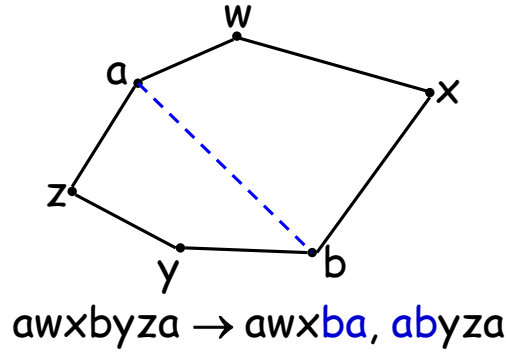


Figure 1: The Split a Face Case.

- **Constructor Case:** (add a bridge) Suppose G and H are connected graphs with planar embeddings and disjoint sets of vertices. Let a be a vertex on a discrete face, γ , in the embedding of G . That is, γ is of the form

$$a \dots a.$$

¹ There is one exception to this rule. If G is a line graph beginning with a and ending with b , then the cycles into which γ splits are actually the same. That's because adding edge $a-b$ creates a simple cycle graph, C_n , that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces. But since this is the only situation in which two faces are actually the same cycle, this exception is better explained in a footnote than mentioned explicitly in the definition.

Similarly, let b be a vertex on a discrete face, δ , in the embedding of H , so δ is of the form

$$b \cdots b.$$

Then the graph obtained by connecting G and H with a new edge, $a-b$, has a planar embedding whose discrete faces are the union of the discrete faces of G and H , except that faces γ and δ are replaced by one new face

$$a \cdots ab \cdots ba,$$

as illustrated in Figure 2.

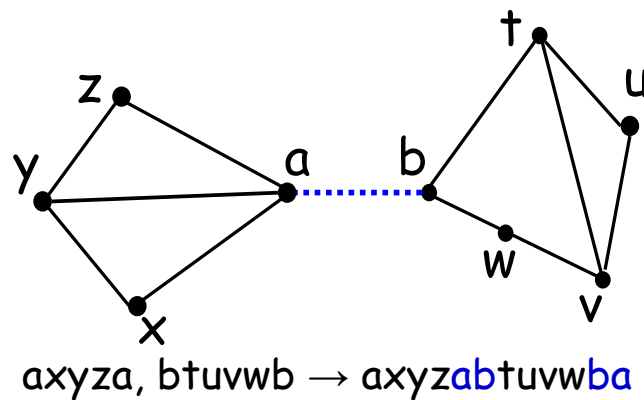


Figure 2: The Add Bridge Case.

An arbitrary graph is *planar* iff each of its connected components has a planar embedding.

Theorem 3.2 (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

Corollary 3.3. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Problem 2.b, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{2}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (2) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

□

Corollary 3.4. K_5 is not planar.

Proof.

$$e = 10 > 9 = 3v - 6.$$

□