In-Class Problems Week 7, Fri.

Problem 1. (Carried over from Monday, March 19.)

(a) Show that if a connected planar graph with more than two vertices is bipartite, then

$$e \leq 2v - 4. \tag{1}$$

Hint: Similar to the proof that $e \leq 3v - 6$ (see the Appendix).

(b) Conclude that that $K_{3,3}$ is not planar. ($K_{3,3}$ is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)

Problem 2. (Carried over from Friday, March 16.)

The Beneš network has a max congestion of 1; that is, every permutation can be routed in such a way that a single packet passes through each switch. Let's work through an example. A Beneš network of size $N = 8$ appears on the last page.

(a) Within the Beneš network of size $N = 8$, there are two subnetworks of size $N = 4$. Put boxes around these. Hereafter, we'll refer to these as the *upper* and *lower* subnetworks.

(b) Now consider the following permutation routing problem:

$$\begin{array}{ll}
\pi(0) = 3 & \qquad \pi(4) = 2 \\
\pi(1) = 1 & \qquad \pi(5) = 0 \\
\pi(2) = 6 & \qquad \pi(6) = 7 \\
\pi(3) = 5 & \qquad \pi(7) = 4
\end{array}$$

Each packet must be routed through either the upper subnetwork or the lower subnetwork. Construct a graph with vertices 0, 1, …, 7 and draw a *dashed* edge between each pair of packets that can not go through the same subnetwork because a collision would occur in the second column of switches.

(c) Add a *solid* edge in your graph between each pair of packets that can not go through the same subnetwork because a collision would occur in the next-to-last column of switches.

(d) Color the vertices of your graph red and blue so that adjacent vertices get different colors. Why must this be possible, regardless of the permutation π?

(e) Suppose that red vertices correspond to packets routed through the upper subnetwork and blue vertices correspond to packets routed through the lower subnetwork. On the attached copy of the Beneš network, highlight the first and last edge traversed by each packet.

(f) All that remains is to route packets through the upper and lower subnetworks. One way to do this is by applying the procedure described above recursively on each subnetwork. However, since the remaining problems are small, see if you can complete all the paths on your own.

Problem 3. (Carried over from Friday, March 16.)

An *input-output tree network* has n inputs and n outputs, where n is a power of 2. Each input is connected to the root of a binary tree with $n/2$ leaves and with edges pointing away from the root. Likewise, each output is connected to the root of a binary tree with $n/2$ leaves and with edges pointing toward the root.

Two edges point from each leaf of an input tree, and each of these edges points to a leaf of an output tree. The matching of leaf edges is arranged so that for every input and output tree, there is an edge from a leaf of the input tree to a leaf of the output tree, and every output tree leaf has exactly two edges pointing to it.

(a) Draw such an input-output tree net for $n = 4$.

(b) Fill in the table for input-output tree nets, and explain your entries.

# switches	switch size	diameter	max congestion

1 Appendix

Theorem 3.1 (Euler's Formula)**.** *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

Lemma 3.2. *In a planar embedding of a graph, each edge is traversed a total of two times by the faces of the embedding.*

Lemma 3.3. *In a planar embedding of a graph with at least three vertices, each face is of length at least three.*

Corollary 3.4. *Suppose a connected planar graph has $v \geq 3$ vertices and e edges. Then*

$$e \leq 3v - 6.$$

Proof. By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with v vertices and e edges has a planar embedding with f faces. By Lemma 3.2, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 3.3, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{2}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (2) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

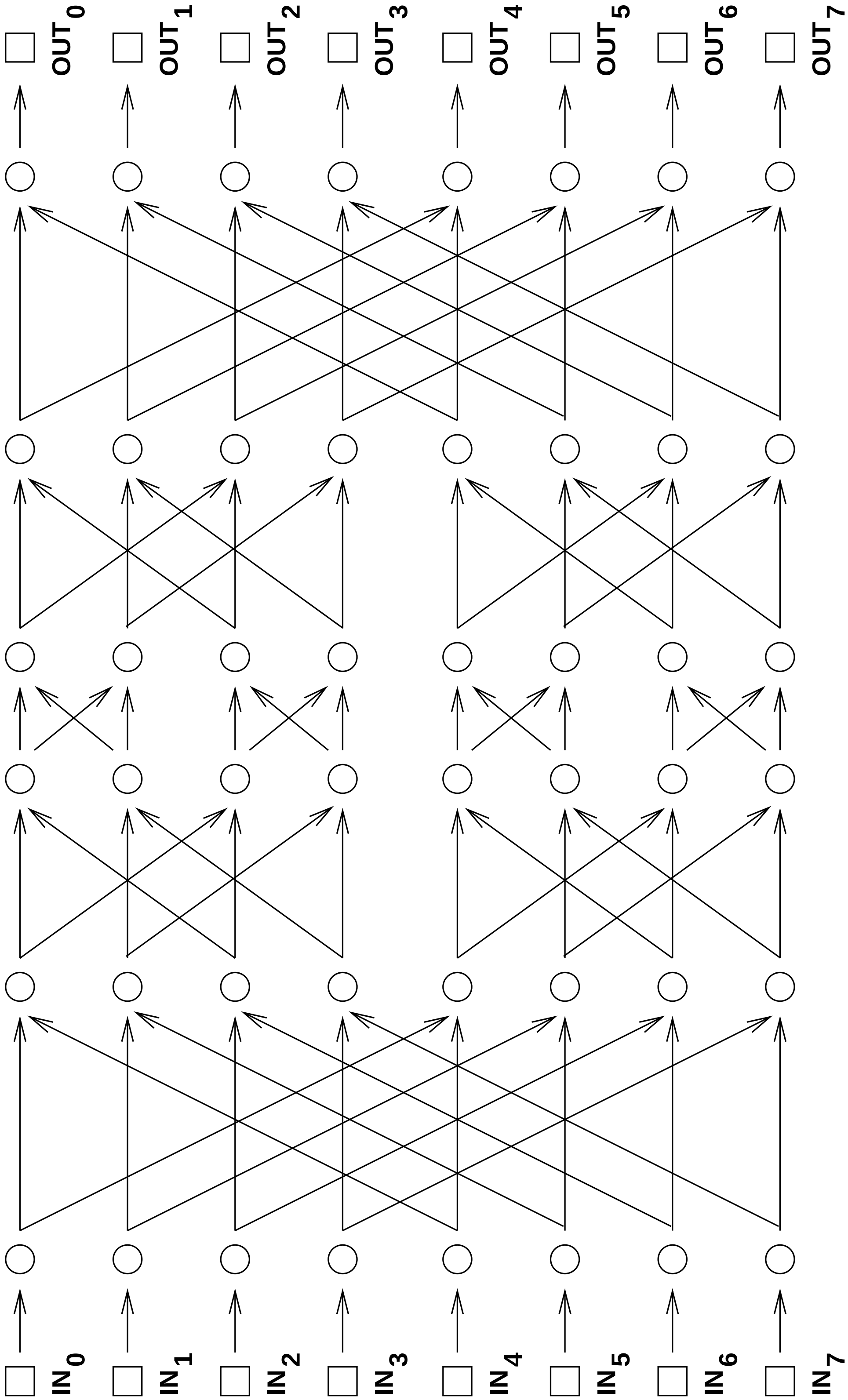
□

Corollary 3.5. *K_5 is not planar.*

Proof.

$$e = 10 > 9 = 3v - 6.$$

□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