In-Class Problems Week 6, Wed.

Problem 1. MIT has a lot of student clubs loosely overseen by the MIT Student Association. Each eligible club would like to delegate one of its members to appeal to the Dean for funding, but the Dean will not allow a student to be the delegate of more than one club. Fortunately, one of the officers of the Association took 6.042 and so is able to guarantee that there is a way for each club to select a delegate to appeal for funds. What the Association officer noticed is that the Association's data shows that no student is a member of more than 9 clubs. The officer also knows that to be eligible for support from the Dean's office, a club must have at least 13 members.

(a) Explain how to model the delegate selection problem as a bipartite matching problem.

(b) Explain why the Student officer can guarantee there is a proper delegate selection. (If only the Association officer had taken 6.046, *Algorithms*, they could, without an excessive computation, even have found a possible delegate selection for all the clubs.)

(c) The Student Association officer used the fact that the condition for a matching to exist, namely, that every set of c clubs had a total of at least c members, was sure to hold because the degree of every club was greater than the degree of every student. See if you can come up with a proof of this fact (without looking it up in the Notes).

Problem 2. Through a series of acquisitions, the CellTel corporation has obtained a nationwide network of cell phone towers. Each tower can support up to 2110 callers, but a CellTel customer can only contact a tower that is within 10 miles. As a consultant to CellTel, your job is to explain why, at any given moment, their network can service all callers iff, at that moment, the following condition holds:

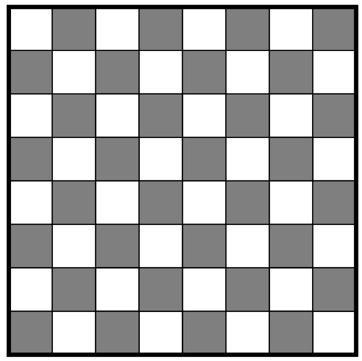
> *The size of any set of callers is at most 2110 times the number of towers within range of at least one caller in the set.* (*)

(a) Describe how to model this situation as a bipartite graph matching problem.

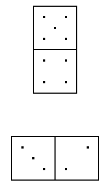
(b) Explain why the network can service all the callers iff condition (*) holds.

Problem 3. Suppose that one domino can cover exactly two squares on a chessboard, either vertically or horizontally.

(a) Can you tile an 8×8 chessboard with 32 dominos?

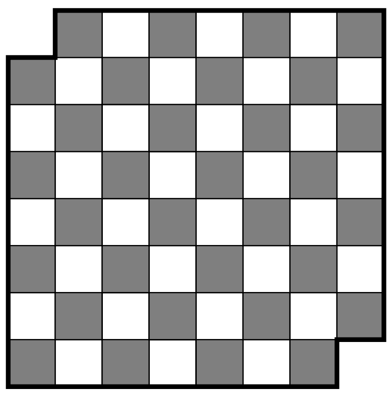


chess board

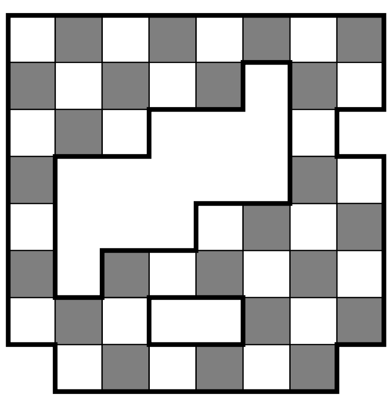


dominos

(b) Can you tile an 8×8 chessboard with 31 dominos if opposite corners are removed?



(c) Now suppose that an assortment of squares are removed from a chessboard. An example is shown below.



Explain why a board can be tiled with dominos iff there are an equal number of black squares and white squares, and every set of white squares is adjacent to at least as many black squares.

Hint: Given a truncated chessboard with an equal number of white and black squares, show how to construct a bipartite graph that has a perfect matching iff the chessboard can be tiled with dominos.

Bipartite Matching

Suppose S is a set of vertices in a graph. Define $N(S)$ to be the set of all neighbors of S; that is, all vertices that are adjacent to a vertex in S. S is called a *bottleneck* if

$$|S| > |N(S)|\,.$$

A *matching* for S is a set of edges such that

- the edges are non-overlapping —no two edges are incident to the same vertex,
- every edge is incident to exactly one vertex in S, and
- every vertex in S is incident to an edge.

Theorem 3.1 (Hall's Theorem). *Let G be a bipartite graph, that is, a graph whose vertices can be separated (partitioned) into two sets, L and R, such that every edge has one endpoint in L and one endpoint in R. There is a matching for L iff no subset of L is a bottleneck.*

Definition 3.2. A bipartite graph G with vertex partition L, R is *degree-constrained* if $\deg(l) \geq \deg(r)$ for every $l \in L$ and $r \in R$.

Lemma 3.3. *Every degree-constrained bipartite graph satisifies the matching condition.*