In-Class Problems Week 5, Wed.

Problem 1. **(a)** Prove that in every graph, there are an even number of vertices of odd degree.

Hint: The Handshaking Theorem.

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

Problem 2. For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for a—b.)

(a)

$$G_1 \text{ with } V_1 = \{1,2,3,4,5,6\},\ E_1 = \{12, 23, 34, 14, 15, 35, 45\}$$
$$G_2 \text{ with } V_2 = \{1,2,3,4,5,6\},\ E_2 = \{12, 23, 34, 45, 51, 24, 25\}$$

(b)

$$G_1 \text{ with } V_1 = \{1,2,3,4,5,6\},\ E_1 = \{12, 23, 34, 14, 45, 56, 26\}$$
$$G_2 \text{ with } V_2 = \{a,b,c,d,e,f\},\ E_2 = \{ab, bc, cd, de, ae, ef, cf\}$$

(c)

$$G_1 \text{ with } V_1 = \{a,b,c,d,e,f,g,h\},\ E_1 = \{ab, bc, cd, ad, ef, fg, gh, he, dh, bf\}$$
$$G_2 \text{ with } V_2 = \{s,t,u,v,w,x,y,z\},\ E_2 = \{st, tu, uv, sv, wx, xy, yz, wz, sw, vz\}$$

Problem 3. A graph is ***connected*** when for every pair of vertices, u and v, there is a path between u and v.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

(a) Prove that this Claim is indeed false by providing a counterexample.

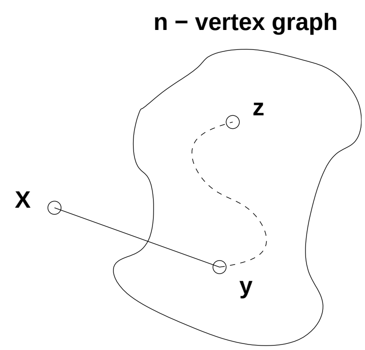
(b) Since the claim is false, there must be an logical mistake in the following "proof." Pinpoint the *first* logical mistake (unjustified step) in the proof.

False proof. We use induction. Let $P(n)$ be the proposition that if every vertex in an n-vertex graph has positive degree, then the graph is connected.

Base cases: ($n \le 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \ge 2$. Consider an n-vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n+1)$-vertex graph:



All that remains is to check that there is a path from x to every other vertex z. Since x has positive degree, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge x—y to the path from y to z. This proves $P(n+1)$.

By the principle of induction $P(n)$ is true for all $n \ge 0$, which proves the theorem.

□

Problem 4. **(a)** For any vertex, v, in a graph, let $\widehat{v}$ be the set of vertices adjacent to v, that is,

$$\widehat{v} ::= \{v' \mid v\text{—}v' \text{ is an edge of the graph}\}.$$

Suppose f is an isomorphism from graph G to graph H. Carefully prove that $f(\widehat{v}) = \widehat{f(v)}$.

(b) Conclude that if G and H are isomorphic graphs, then for each $k \in \mathbb{N}$, they have the same number of degree k vertices.

Definitions

A *path* in a graph, G, is a sequence of $k \geq 0$ vertices

$$v_0, \ldots, v_k$$

such that v_i—v_{i+1} is an edge of G for all i where $0 \leq i < k$. The path is said to *start* at v_0, to *end* at v_k, and *length* of the path is defined to be k. An edge, e, is ***traversed*** n ***times*** by the path if there are n different values of i such that edge v_i—v_{i+1} is e.

The path is *simple* iff all the v_i's are different, that is, $v_i = v_j$ only if $i = j$.

Two vertices in a graph are *connected* iff there is a path that begins with one of the vertices and ends with the other. The *shortest* path between two vertices is always simple.

A graph is *connected* iff every pair of vertices is connected.

Cycles are paths that begin and end with the same vertex. *Simple* cycles are cycles that don't cross themselves.[1]

[1]The technical definition of simple cycle appears in Week 5 Notes.