In-Class Problems Week 4, Mon.

Problem 1. Provide simple *recursive* definitions of the following sets:

(a) The set $S ::= \{2^k 3^m 5^n \mid k, m, n \in \mathbb{N}\}$.

(b) The set $T ::= \{2^k 3^{2k+m} 5^{m+n} \mid k, m, n \in \mathbb{N}\}$.

(c) The set $L ::= \{(a, b) \in \mathbb{Z}^2 \mid a + 2b = 3k \text{ for some } k \in \mathbb{Z}\}$.

Problem 2. The Elementary 18.01 Functions (F18's) are the set of functions of one real variable defined recursively as follows:

Base cases:

1. The identity function, $\mathrm{id}(x) ::= x$ is an F18,
2. any constant function is an F18,
3. the sine function is an F18,

Constructor cases:

If f, g are F18's, then so are

1. $f + g$, fg, e^g (the constant e),
2. the inverse function f^{-1},
3. the composition $f \circ g$.

Prove, by Structural Induction on this definition, that the Elementary 18.01 Functions are *closed under taking derivatives*. That is, show that if f is an F18, then so is df/dx.

Problem 3. BAexp's are defined in the Appendix.

(a) The value of flatten(e) for $e \in$ BAexp is the sequence of integers in e obtained by "erasing" everything but the integers that appear within tagged variables and tagged `int`'s. For example,

$$\begin{aligned} e &::= \langle \texttt{sum}, \langle \texttt{var}, 3\rangle, \langle \texttt{sum}, \langle \texttt{var}, 2\rangle, \langle \texttt{int}, 2\rangle\rangle\rangle \\ f &::= \langle \texttt{prod}, \langle \texttt{var}, 4\rangle, \langle \texttt{var}, 5\rangle\rangle \\ g &::= \langle \texttt{prod}, e, \langle \texttt{sum}, \langle \texttt{var}, 7\rangle, f\rangle\rangle \\ \text{flatten}(g) &= \langle 3, 2, 2, 7, 4, 5\rangle . \end{aligned}$$

Give a recursive definition of flatten. (You may use the operation of *concatenation* (append) of two sequences.)

(b) Prove by structural induction on the definition of BAexp that for all $e \in$ BAexp,

$$2 \cdot \text{length}(\text{flatten}(e)) = |e| + 1$$

Appendix

The set, BAexp, of *Basic Arithmetic Expressions* is defined recursively as a tagged data type as follows:

- **Base cases:** If $n \in \mathbb{Z}$, then
  1. $\langle \texttt{int}, n\rangle \in$ BAexp, and
  2. $\langle \texttt{var}, n\rangle \in$ BAexp.
- **Constructor cases:** if $e, e' \in$ BAexp, then
  1. $\langle \texttt{sum}, e, e'\rangle \in$ BAexp, and
  2. $\langle \texttt{prod}, e, e'\rangle \in$ BAexp.

The size, $|e|$, of $e \in$ BAexp is defined recursively on this definition by:

- **Base cases:**
  1. $|\langle \texttt{int}, n\rangle| ::= 1$
  2. $|\langle \texttt{var}, n\rangle| ::= 1$
- **Constructor cases:**
  1. $|\langle \texttt{sum}, e_1, e_2\rangle| ::= |e_1| + |e_2| + 1$
  2. $|\langle \texttt{prod}, e_1, e_2\rangle| ::= |e_1| + |e_2| + 1$