In-Class Problems Week 4, Fri.

Problem 1. Define *lexicographic order*, $\prec_{\text{lex}}$, on $\mathbb{N}^2$ by the rule:

$$(a_2, b_2) \prec_{\text{lex}} (a_1, b_1)$$

iff

$$a_2 < a_1 \text{ or } [a_2 = a_1 \text{ and } b_2 < b_1].$$

Prove that there is *no* infinite $\prec_{\text{lex}}$-decreasing sequence

$$(a_1, b_1) \succ_{\text{lex}} (a_2, b_2) \succ_{\text{lex}} \cdots \succ_{\text{lex}} (a_n, b_n) \succ_{\text{lex}} \cdots \tag{1}$$

Hint: Consider the smallest a_m.

Problem 2. Consider the following game for two players. The players alternate moves. A move consists of a pair (x, y) of natural numbers, subject to the constraint that none of the previous moves may simultaneously be below and to the left of the current move. That is, if (x, y) is the current move and (x', y') is any previous move, then either $x < x'$ (so the previous move is to the right of the current one) or $y < y'$ (so the previous move is above the previous one). A player who moves to the origin $(0, 0)$ is the loser.

For example, the Player 1 might choose (5,6), after which Player 2 can move to any point (n, m) such that $n < 5$ or $m < 6$, for example, (4,12). Now the players might move successsively to (4,11), (29,5), (1,1), (0,54), (0,1). At this point it's Player 2's turn, and he can move to (1,0). At this point it is Player 1's move, and the only available move is to the origin (0,0), so Player 1 loses this play of the game.

(a) Identify a winning strategy for the first player, and argue its correctness. Does your strategy guarantee any bound on the number of game moves?

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(b) Even if the players conspire to keep the game going as long as possible, it will necessarily terminate. Prove this as follows:

i. At any point in the game, let x_m be the minimum of the x coordinates of all of the previous moves, and likewise, y_m be the minimum of the y coordinates of all of the previous moves. Verify that $x_m + y_m$ is a weakly decreasing natural-number valued variable.

ii. Suppose a is the least number such that a move (x_m, a) has been made, and likewise b is least such that move (b, y_m) has been made. The *bounded moves* are defined to be the possible moves in the rectangle with corners at (x_m, a) and (b, y_m). Explain why the only moves that do not decrease $x_m + y_m$ must be bounded moves.

iii. Define the *size*, σ, of a game position to be $(x_m + y_m, k)$ where k is the number of bounded moves. Explain why σ is a strictly decreasing variable under lexicographic order, and use this to conclude that the game must terminate.

(c) *(Optional)* Is there a winning strategy for the first player that guarantees a bound on the number of game moves?

(d) Conclude that under the coordinatewise partial order, $\mathbb{N}^2$ has no infinite antichain.

(e) In $\mathbb{N}^2$ under the coordinatewise partial order, describe an antichain of size n, for each $n > 1$.