In-Class Problems Week 3, Tue.

Problem 1. A pair of 6.042 TAs, Chiyoun and Tina, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Tina's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Chiyoun's cat, Tailspin - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3.
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

(a) Give some valid order in which the tasks might be completed.

Chiyoun and Tina want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Chiyoun cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

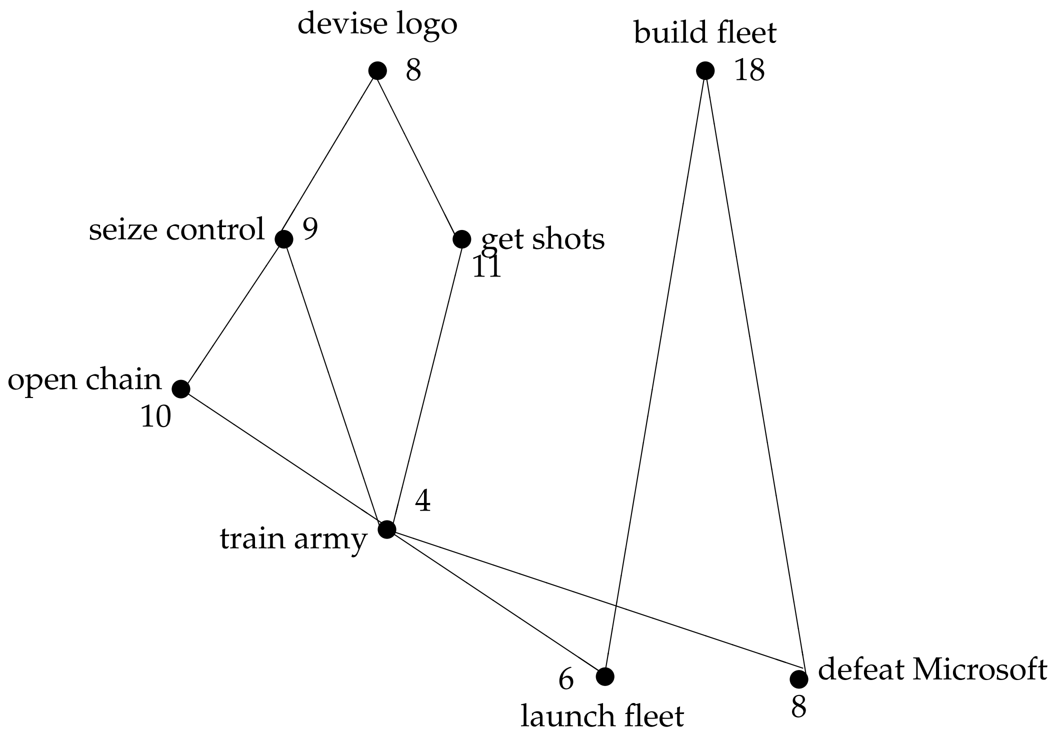

Figure 1: Graph representing the task precedence constraints.

(b) Chiyoun and Tina want to know how long conquering the galaxy will take. Tina suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

(c) Chiyoun proposes a different method for determining the duration of their project. He suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

(d) What is the minimum number of days that Chiyoun and Tina need to conquer the galaxy? No proof is required.

Problem 2. Verify that each of the following relations is a partial order. For each, indicate if it is strict, weak, and/or a total order. (Definitions are in the Appendix.)

(a) The superset relation, $\supseteq$, on some family of sets.

(b) The "divides" relation on natural numbers.

Problem 3. **(a)** What are the maxim*al* and minim*al* elements, if any, of the set, $\mathbb{N}$, of all nonnegative integers under divisibility? Is there a minim*um* or maxim*um* element?

(b) What are the minimal and maximal elements, if any, of the set of integers ≥ 2 under divisibility?

(c) What is the size of the longest chain that is guaranteed to exist in any partially ordered set of $n > 0$ elements? What about the largest antichain?

(d) Describe a partially ordered set that has no minimal or maximal elements.

(e) Describe a partially ordered set that has a *unique minimal* element, but no minimum element. *Hint:* It will have to be infinite.

Problem 4. Prove that a binary relation, R, on a set, A, is a strict partial order iff it is transitive and irreflexive.

Appendix

Relational Properties

A binary relation, R, on a set, A, is

- *transitive* if for every $a, b, c \in A$, aRb and bRc implies aRc.
- *asymmetric* if for every $a, b \in A$, aRb implies $\neg(bRa)$,
- *reflexive* if aRa for every $a \in A$,
- *antisymmetric* if for every $a \neq b \in A$, aRb implies $\neg(bRa)$,
- *irreflexive* if aRa holds for no $a \in A$.

Partial Order

A binary relation is a *strict partial order* iff it is transitive and asymmetric. It is a *weak partial order* iff it is transitive, reflexive, and antisymmetric.

Let $\preceq$ be a weak (reflexive) partial order on a set, A.

- An element $a \in A$ is *minimal* iff there is no element in A that is $\preceq a$ except possibly a itself. Similarly, an element $a \in A$ is *maximal* iff there is no element in A that is $\succeq a$ except possibly a itself.
- An element $a \in A$ is a *lower bound* for a subset, $S \subseteq A$ iff $a \preceq s$ for every $s \in S$. Similarly, an element $a \in A$ is an *upper bound* for a subset, $S \subseteq A$ iff $s \preceq a$ for every $s \in S$.
- An element $a \in A$ is the *minimum* element iff a is a lower bound on A. Similarly, an element $a \in A$ is the *maximum* element iff a is an upper bound on A.
- Elements $a, b \in A$ are *comparable* iff either $a \preceq b$ or $b \preceq a$. Two elements are *incomparable* iff they are not comparable.
- A subset, $S \subseteq A$ is *totally ordered* iff every two distinct elements in S are comparable.
- A *chain* is a totally ordered subset of A.
- An *antichain* is a subset of A, such that no two elements in it are comparable.