In-Class Problems Week 14, Mon.

Problem 1. A herd of cows is stricken by an outbreak of *cold cow disease*. The disease lowers the normal body temperature of a cow, and a cow will die if its temperature goes below 90 degrees F. The disease epidemic is so intense that it lowered the average temperature of the herd to 85 degrees. Body temperatures as low as 70 degrees, **but no lower**, were actually found in the herd.

(a) Based solely on the information above, use Markov's bound to state an upper bound on the probability that a randomly chosen cow from the herd will have a high enough temperature to survive. Try to make the bound as small as possible.

(b) Suppose there are 400 cows in the herd. Give an example set of temperatures for the cows so that the probability that a randomly chosen cow will have a high enough temperature to survive is as large as possible, and explain why it cannot be larger.

Problem 2. A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability $1/6$, a hand of black jack with probability $1/2$, and a hand of stud poker with probability $1/5$. Assume the outcomes of the card games are mutually independent.

(a) What is the expected number of hands the gambler wins in a day?

(b) What would the Markov bound be on the probability that the gambler will win at least 108 hands on a given day?

(c) What is the variance in the number of hands won per day?

(d) What would the Chebyshev bound be on the probability that the gambler will win at least 108 hands on a given day? You may answer with a numerical expression that is not completely evaluated.

Problem 3. The hat-check staff has had a long day serving at a party, and at the end of the party they simply return people's hats at random. Assume that n people checked hats at the party.

(a) What is the expected number of people who get their own hat back?

Let $X_i = 1$ be the indicator variable for the ith person getting their own hat back. Let $S_n = \sum_{i=1}^{n} X_i$, so S_n is the total number of people who get their own hat back.

(b) Write a simple formula for $\text{E}[X_i X_j]$ for $i \neq j$. *Hint:* What is $\Pr\{X_j = 1 \mid X_i = 1\}$?

(c) Explain why you cannot use the variance of sums formula to calculate $\text{Var}[S_n]$.

(d) Show that $\text{E}\left[S_n^2\right] = 2$. *Hint:* $X_i^2 = X_i$.

(e) What is the variance of S_n?

(f) Use Chebyshev's bound to show that the probability that 11 or more people get their own hat back is at most 0.01.

Problem 4. This problem is a review of the derivation of Chebyshev's Theorem from Markov's Theorem.

(a) Explain why the following corollary of Markov's Theorem holds:

Corollary. *For any random variable R, any positive integer k, and any $x > 0$,*

$$\Pr\{|R| \geq x\} \leq \frac{\text{E}\left[|R|^k\right]}{x^k}.$$

(b) Use the above corollary to prove the following:

Theorem (Chebyshev)**.** *Let R be a random variable, and let x be a positive real number. Then*

$$\Pr\{|R - \text{E}[R]| \geq x\} \leq \frac{\text{Var}[R]}{x^2}.$$

Problem 5. **(a)** Prove from the definition of variance (in the Appendix) that

$$\text{Var}[R] = \text{E}\left[R^2\right] - \text{E}^2[R].$$

(b) Prove that $\text{Var}[X + Y + Z] = \text{Var}[X] + \text{Var}[Y] + \text{Var}[Z]$, if X, Y, Z are *pairwise* independent.

Appendix

The *expectation* of a random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}$$

Theorem (Markov's Theorem)**.** *If R is a nonnegative random variable, then for all $x > 0$*

$$\Pr\{R \geq x\} \leq \frac{\mathrm{E}[R]}{x}.$$

The *variance*, $\mathrm{Var}[R]$, of a random variable, R, is:

$$\mathrm{Var}[R] ::= \mathrm{E}\left[(R - \mathrm{E}[R])^2\right].$$

It is easy to show that

$$\mathrm{Var}[R] = \mathrm{E}\left[R^2\right] - \mathrm{E}^2[R].$$

[Variance of an index variable], I, with $\Pr\{I = 1\} = p$:

$$\mathrm{Var}[I] = pq$$

where $q ::= 1 - p$.

[Variance and constants] For constants, a, b,

$$\mathrm{Var}[aR + b] = a^2 \,\mathrm{Var}[R].$$

[Variance Additivity] If $R_1, R_2, \ldots, R_n$ are *pairwise* independent variables, then

$$\mathrm{Var}[R_1 + R_2 + \cdots + R_n] = \mathrm{Var}[R_1] + \mathrm{Var}[R_2] + \cdots + \mathrm{Var}[R_n]$$

Theorem (Chebyshev)**.** *Let R be a random variable, and let x be a positive real number. Then*

$$\Pr\{|R - \mathrm{E}[R]| \geq x\} \leq \frac{\mathrm{Var}[R]}{x^2}.$$