In-Class Problems Week 13, Mon.

Problem 1. Suppose $H_{n,p}$ and $H_{m,p}$ are independent binomially distributed random variables. What is $\Pr\{H_{n,p} + H_{m,p} = k\}$?

Problem 2. **(a)** Prove that

$$\Pr\left\{H_{2n,1/2} = n\right\} \sim \frac{1}{\sqrt{\pi n}}. \tag{1}$$

Hint: Use Stirling's approximation (in the appendix). Note that the entropy function $H(\alpha)$ is 1 for $\alpha = 1/2$.

(b) Estimate the probability that the number of heads in 400 flips of a fair coin will be between 195 and 205, and likewise that in 10,000 flips it will be between 4980 and 5020.

Also, discuss writing a program to calculate the exact answer.

Appendix

Stirling's Approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Binomial Variables

A random variable, J, is (n,p)-*binomial* for $n \in \mathbb{N}$ and $0 < p < 1$, if

$$J = \sum_{k=1}^{n} H_k$$

where $H_1, H_2, \ldots, H_n$ are mutually independent indicator variables with $\Pr\{H_i = 1\} = p$ for all i. Equivalently, J is (n,p)-*binomial* iff PDF_J has the (n,p)-*binomial distribution*:

$$\mathrm{PDF}_J(k) ::= \binom{n}{k} p^k (1-p)^{n-k},$$

for $0 \le k \le n$.

Binomial bounds

If J is an (n,p)-binomial variable, the following formula gives a fairly tight upper bound on PDF_J.

$$\mathrm{PDF}_J(\alpha n) \le \frac{2^{nH(\alpha)}}{\sqrt{2\pi\alpha(1-\alpha)n}} \cdot p^{\alpha n}(1-p)^{(1-\alpha)n} \tag{2}$$

where H is the entropy function,

$$H(\alpha) ::= -\left(\alpha \log_2 \alpha + (1-\alpha)\log_2(1-\alpha)\right).$$

The bounding formula is also asymptotically equal to PDF_J.