In-Class Problems Week 13, Fri.

Problem 1. Here are seven propositions:

$$\begin{array}{rcrcr}
x_1 & \vee & x_3 & \vee & \neg x_7 \\
\neg x_5 & \vee & x_6 & \vee & x_7 \\
x_2 & \vee & \neg x_4 & \vee & x_6 \\
\neg x_4 & \vee & x_5 & \vee & \neg x_7 \\
x_3 & \vee & \neg x_5 & \vee & \neg x_8 \\
x_9 & \vee & \neg x_8 & \vee & x_2 \\
\neg x_3 & \vee & x_9 & \vee & x_4
\end{array}$$

Note that:

1. Each proposition is the disjunction (OR) of three terms of the form x_i or the form $\neg x_i$.
2. The variables in the three terms in each proposition are all different.

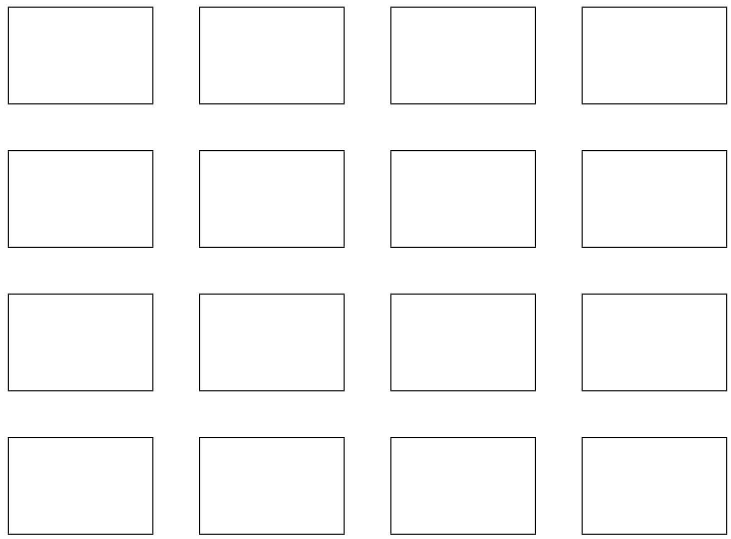
Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability.

(a) What is the expected number of true propositions?

Hint: Let T_i be an indicator for the event that the i-th proposition is true.

(b) Use your answer to prove that for *any* set of 7 propositions satisfying the conditions 1. and 2., there is an assignment to the variables that makes all 7 of the propositions true.

Problem 2. A classroom has sixteen desks arranged as shown below.



If there is a girl in front, behind, to the left, or to the right of a boy, then the two of them *flirt*. One student may be in multiple flirting couples; for example, a student in a corner of the classroom can flirt with up to two others, while a student in the center can flirt with as many as four others. Suppose that desks are occupied by boys and girls with equal probability and mutually independently. What is the expected number of flirting couples?

Problem 3. Let R_1 and R_2 be random variables on a sample space, $\mathcal{S}$.

(a) Prove that
$$\mathrm{E}[R_1 + R_2] = \mathrm{E}[R_1] + \mathrm{E}[R_2].$$

Hint:
$$\mathrm{E}[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \Pr\{\omega\} \tag{1}$$

(b) Justify each line of the following proof that if R_1 and R_2 are *independent*, then
$$\mathrm{E}[R_1 \cdot R_2] = \mathrm{E}[R_1] \cdot \mathrm{E}[R_2].$$

Proof.

$$\begin{aligned}
&\mathrm{E}[R_1 \cdot R_2] \\
&= \sum_{r \in \mathrm{range}(R_1 \cdot R_2)} r \cdot \Pr\{R_1 \cdot R_2 = r\} \\
&= \sum_{r_i \in \mathrm{range}(R_i)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ and } R_2 = r_2\} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \sum_{r_2 \in \mathrm{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1 \text{ and } R_2 = r_2\} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \sum_{r_2 \in \mathrm{range}(R_2)} r_1 r_2 \cdot \Pr\{R_1 = r_1\} \cdot \Pr\{R_2 = r_2\} \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} \left(r_1 \Pr\{R_1 = r_1\} \cdot \sum_{r_2 \in \mathrm{range}(R_2)} r_2 \Pr\{R_2 = r_2\} \right) \\
&= \sum_{r_1 \in \mathrm{range}(R_1)} r_1 \Pr\{R_1 = r_1\} \cdot \mathrm{E}[R_2] \\
&= \mathrm{E}[R_2] \cdot \sum_{r_1 \in \mathrm{range}(R_1)} r_1 \Pr\{R_1 = r_1\} \\
&= \mathrm{E}[R_2] \cdot \mathrm{E}[R_1].
\end{aligned}$$

□

Problem 4. **(a)** Compute the expected value of the number rolled on a fair, six-sided die, given that the outcome is even.

(b) Define the random variable R using the following procedure. Roll two fair, independent dice and let the numbers rolled be D_1 and D_2. Then pick a random card from a standard deck. If the suit of the card is spades, let the value of R be D_1. Otherwise, let R be $D_1 \cdot D_2$. What is the expected value of R?

Appendix

For random variable, R,

$$\mathrm{E}[R] ::= \sum_{x \in \text{range}(R)} x \cdot \Pr\{R = x\}.$$

If R is defined on a sample space, $\mathcal{S}$, then

$$\mathrm{E}[R] = \sum_{\omega \in \mathcal{S}} R(\omega) \Pr\{\omega\}.$$

The expected value of a random variable, R, *given event* A, is

$$\mathrm{E}[R \mid A] ::= \sum_{x \in \text{range}(R)} x \cdot \Pr\{R = x \mid A\}$$

So $\mathrm{E}[R] = \mathrm{E}[R \mid \mathcal{S}]$ where $\mathcal{S}$ is the sample space for R.

[The Law of Total Expectation] Let $A_1, A_2, \ldots$ be a partition of the sample space. Then

$$\mathrm{E}[R] = \sum_i \mathrm{E}[R \mid A_i] \Pr\{A_i\}.$$