In-Class Problems Week 12, Wed.

Problem 1. There are two decks of cards. One is complete, but the other is missing the ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

Problem 2. There is a rare and deadly disease called *Nerditosis* which afflicts about 1 person in 1000. One symptom is a compulsion to refer to everything— fields of study, classes, buildings, etc.— using numbers. It's horrible. As victims enter their final, downward spiral, they're awarded a degree from MIT. Two doctors claim that they can diagnose Nerditosis.

(a) Doctor X received his degree from Harvard Medical School. He practices at Massachusetts General Hospital and has access to the latest scanners, lab tests, and research. Suppose you ask Doctor X whether you have the disease.

- If you have Nerditosis, he says "yes" with probability 0.99.
- If you don't have it, he says "no" with probability 0.97.

Let D be the event that you have the disease, and let E be the event that the diagnosis is erroneous. Use the Total Probability Law to compute $\Pr\{E\}$, the probability that Doctor X makes a mistake.

(b) "Doctor" Y received his genuine degree from a fully-accredited university for $49.95 via a special internet offer. He knows that Nerditosis stikes 1 person in 1000, but is a little shaky on how to interpret this. So if you ask him whether you have the disease, he'll randomly say "yes" with probability 1 in 1000 without even examining you.

Let D be the event that you have the disease, and let F be the event that the diagnosis is faulty. Use the Total Probability Law to compute $\Pr\{F\}$, the probability that Doctor Y made a mistake.

(c) Which doctor is more reliable?

(d) Doctor Z, who went to MIT and took 6.042, observes that he can do even better than Doctor Y. How?

Problem 3. There were n Immortal Warriors born into our world, but in the end *there can be only one*. The Immortals' original plan was to stalk the world for centuries, dueling one another with ancient swords in dramatic landscapes until only one survivor remained. However, after a thought-provoking discussion of probabilistic independence, they opt to give the following protocol a try:

1. The Immortals forge a coin that comes up heads with probability p.
2. Each Immortal flips the coin once.
3. If *exactly one* Immortal flips heads, then he or she is declared The One. Otherwise, the protocol is declared a failure, and they all go back to hacking each other up with swords.

(a) One of the Immortals (Kurgan from the Russian steppe) argues that as n grows large, the probability that this protocol succeeds must tend to zero. Another (McLeod from the Scottish highlands) argues that this need not be the case, provided p is chosen *very carefully*. What does your intuition tell you?

(b) What is the probability that the experiment succeeds as a function of p and n?

(c) How should p, the bias of the coin, be chosen in order to maximize the probability that the experiment succeeds? (You're going to have to compute a derivative!)

(d) What is the probability of success if p is chosen in this way? What quantity does this approach when n, the number of Immortal Warriors, grows large?

Problem 4. Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \leq i \leq n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \leq \Pr\{F\} \leq np. \tag{1}$$

We may as well assume $p < 1/n$, since the upper bound is trivial otherwise. For example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \ldots, n\})$ of subsets of positive integers $\leq n$, where $s \in \mathcal{S}$ corresponds to the indices of the components which fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the sample points.

(b) Show that the probability that the system fails could actually be as large as np by describing appropriate probabilities for the sample points.

(c) Prove the inequality (1).

1 Appendix

The Total Probability Law is

$$\Pr\{A\} = \Pr\{A \mid E\} \cdot \Pr\{E\} + \Pr\{A \mid \overline{E}\} \cdot \Pr\{\overline{E}\}.$$