In-Class Problems Week 12, Fri.

Problem 1. For the game of "guess the bigger integer from 0 to 7" described in lecture, give a strategy for Team 1 (the team that picks the integers that go on each piece of paper) which guarantees that $\Pr\{\text{Team 1 loses}\} \leq 4/7$.

For this purpose, assume Team 2 (the team that decides to stick or switch) plays by a probabilistic strategy: if the integer they see is k, they switch with probability p_k, for $k = 0, \dots 7$. Use the 4-step tree approach to define the sample space and the probability of each outcome.

Problem 2. **(a)** Prove that if events E and F are independent then so are $\overline{E}$ and F.

(b) For any event A, its *indicator variable*, I_A, is the 0-1 valued variable such that the event $[I_A = 1]$ is the same as the event A. It follows that $[I_A = 0]$ is the same as $\overline{A}$. Prove that two events E, F are independent iff their indicator random variables I_E, I_F are independent. (The definitions of independence for events and for random variables are not the same, so there is some proving to do.)

(c) How about for three events E, F, G?

Problem 3. Independently flip three fair coins ("fair" means equally likely to come up with a head or a tail), and let H_i be the indicator variable for a head occurring on the ith flip, for $i = 1, 2, 3$. Define $C ::= H_1 + H_2 + H_3$ to be the number of heads flipped, M to be the indicator variable for the event $[H_1 = H_2 = H_3]$ that all three coins match, and S be the indicator variable for the event $[C \equiv 1 \mod 2]$ that an odd number of heads are flipped.

(a) Verify that none of these six variables is independent of C.

(b) Verify that H_1, H_2, H_3, and S are 3-wise independent, but not mutually independent.

(c) Verify that H_1, S and M are not mutually independent.

Appendix

Random Variables

A (real-valued) *random variable* over a given sample space, $\mathcal{S}$, is a total function from $\mathcal{S} \to \mathbb{R}$.

Random variables $R_1, R_2, \ldots$ are ***mutually independent*** iff

$$\Pr\left\{\bigcap_i [R_i = x_i]\right\} = \prod_i \Pr\{R_i = x_i\},$$

for all $x_1, x_2, \cdots \in \mathbb{R}$. They are k***-wise independent*** iff $\{R_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$. Variables that are 2-wise independent are also called ***pairwise independent***.

The Four-Step Method

This is a good approach to questions of the form, "What is the probability that ——-?" Intuition *will* mislead you, but this formal approach gives the right answer every time.

1. Find the sample space. (Use a tree diagram.)
2. Define events of interest. (Mark leaves corresponding to these events.)
3. Determine outcome probabilities:
   (a) Assign edge probabilities.
   (b) Compute outcome probabilities. (Multiply along root-to-leaf paths.)
4. Compute event probabilities. (Sum the probabilities of all outcomes in the event.)